Thm: $\left\{\begin{array}{l}\text{Elliptic fns} \\ \text{wrt } \Lambda\end{array}\right\} = \mathbb{C}(\wp(z), \wp'(z))$.

$\wp(z)$: even fn; $\wp'(z)$: odd fn, so it is necessary.

Lemma: $\forall z_0 \in \mathbb{C}$, $\text{ord}_{z_0}(f) :=$ lowest degree in Laurent series.

if f elliptic, $\displaystyle\sum_{z_0 \in \mathbb{C}/\Lambda} \text{ord}_{z_0}(f) = 0$.
(zero or pole)

Pf use res-thm w.

$$\frac{1}{2\pi i}\int_{\substack{\gamma \\ \text{circle}}} \frac{f'(z)}{f(z)}\,dz = \text{ord}_{z_0}(f)$$

integrate on boundary to get sum, and cancel opposite boundaries. □

pf of thm Basically create a rational expression in $\wp$ & $\wp'$ with same zeroes & poles as an elliptic f. (for even f, for general f split it as even + odd).

Goal: find an algebraic dependence (if any) between $\wp(z)$ & $\wp'(z)$.

$(\wp'(z))^2$ has order 6 poles @ lattice pts, order 2 zeroes at half-lattice pts.

$$(\wp'(z))^2 = c\left(\wp(z) - \wp\left(\tfrac{w_1}{2}\right)\right)\left(\wp(z) - \wp\left(\tfrac{w_2}{2}\right)\right)\left(\wp(z) - \wp\left(\tfrac{w_1+w_2}{2}\right)\right)$$

where $\Lambda = \langle w_1, w_2\rangle$.

$\}$

$$y^2 = c\,(x-e_1)(x-e_2)(x-e_3)$$

$$\mathbb{C}(\wp(z), \wp'(z)) = \mathrm{Frac}\left(\mathbb{C}[x,y]/\langle\quad\rangle\right).$$

Goal: $\mathbb{C}/\Lambda \longleftrightarrow \{(x,y) : y^2 = c(x-e_1)(x-e_2)(x-e_3)\}.$