An elliptic curve is a (smooth proj algebraic) curve of genus 1 with a specified base point.

algebraic defn for genus:

"geometric genus" of $f(x,y) = 0$

is $\frac{(d-1)(d-2)}{2}$ $(d = \deg f)$.

Def A lattice $\Lambda \subset \mathbb{C}$ is a discrete subgp gen by 2 linearly independent elements w_1, w_2.

$\Lambda = \mathbb{Z} w_1 \oplus \mathbb{Z} w_2 = \langle w_1, w_2 \rangle$.

eg $w_1 = 1$, $w_2 = i$.

Def An elliptic fn wrt Λ is a meromorphic fn $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(z+w) = f(z)$ $\forall w \in \Lambda$.

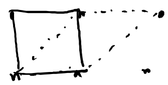
Ex ① constant fn

② any holomorphic elliptic fn is constant.

Pf the image is cpt so it is a singleton.

fundamental parallelogram = fundamental domain = $D = \{aw_1 + bw_2 : a, b \in [0,1)\}$

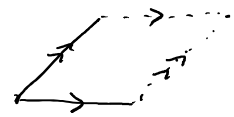
- many options:



(both are valid).

- $D \simeq \mathbb{C}/\Lambda$ is a group

- the values agree on boundaries of an elliptic fn f



by gluing, it follows that f:  $\longrightarrow \mathbb{C}$.

Ex ③ $\wp(z) = \wp(z, \Lambda)$

\wp

$$:= \frac{1}{z^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

(Weierstrass $\wp$-function).

is an elliptic fn.

④ $\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}$

is also an elliptic fn.

⑤ Any polynomial expression in $\wp$ and its derivatives is elliptic.

In fact, we can also do rational expressions.

Q: can we classify all elliptic fns wrt Λ?

Thm $\{$ell fns wrt $\Lambda\} = \mathbb{C}(\wp(z,\Lambda), \wp'(z,\Lambda))$.