Lec 9/4

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$$\mathcal{H} = \{\emptyset\} \cup \{(a,b] \mid -\infty \le a < b < \infty\} \cup \{(a,\infty) \mid -\infty \le a < \infty\}$$

$F : \mathbb{R} \to \mathbb{R}$ increasing $(s \le t \Rightarrow F(s) \le F(t))$

right-cts $(x_n \searrow x \Rightarrow F(x_n) \to F(x))$

extend to $F(-\infty) = \lim_{x \to -\infty} F(x)$

$F(\infty) = \lim_{x \to \infty} F(x)$

$\mu_0(\emptyset) = 0$

$\mu_0(a,b] = F(b) - F(a)$

$\mu_0(a,\infty) = F(\infty) - F(a)$.

$\mathcal{A} := \{$finite disjoint unions of elts of $\mathcal{H}\}$

$\mathcal{A}$ is an algebra.

extend μ_0 to $\mathcal{A}$ by $\mu_0\left(\bigsqcup_{j=1}^{n} H_j\right) = \sum_{j=1}^{n} \mu_0(H_j)$

this is well-defined.

and $m(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$.

<u>Theorem</u>: μ_0 is a premeasure on $\mathcal{A}$.

pf: suppose $(E_n) \subset \mathcal{A}$ is a seq. of disjoint sets & $\coprod E_n \in \mathcal{A}$. Then $\exists$ disjoint h-intervals $F_1 \dots F_k \in \mathcal{H}$ s.t. $\coprod E_n = \coprod_{j=1}^{k} F_j$.

$\forall n$, $E_n = \coprod \underbrace{E_n \cap F_j}_{=: E_n^j}$. $\quad (F_j = \coprod_n E_n^j)$

<u>claim</u>: $\forall j$, $\mu_0(F_j) = \sum_n \mu_0(E_n^j)$.

using the claim & finite additivity of μ_0, $\mu_0(\coprod E_n) = \mu_0(\coprod_{j=1}^{k} F_j) = \sum_{j=1}^{k} \mu_0(F_j)$

$$= \sum_{j=1}^{k} \sum_{n=1}^{\infty} \mu_0(E_n^j)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{k} \mu_0(E_n^j)$$

$$= \sum_{n=1}^{\infty} \mu_0\left(\coprod_{j=1}^{k} (E_n \cap F_j)\right)$$

$$= \sum_{n=1}^{\infty} \mu_0(E_n).$$

pf of Claim:

Step 1: Suppose $(a,b] = \coprod_1^\infty (a_j, b_j]$ $\quad -\infty < a < b < \infty$

Then $\forall n \in \mathbb{N}$, $\coprod_1^n (a_j, b_j] \subset \coprod_1^\infty (a_j, b_j] = (a,b]$.

By monotonicity,

$$\sum_1^n \mu_0 (a_j, b_j] = \mu_0 \left(\coprod_1^n (a_j, b_j]\right) \leq \mu_0 (a,b].$$

$\Rightarrow$ $\sum_1^\infty \mu_0 (a_j, b_j] \leq \mu_0 (a,b]$.

For the other direction, let $\varepsilon > 0$.

- Since F is right cts, $\exists \delta > 0$ s.t.

$$F(a+\delta) - F(a) < \frac{\varepsilon}{2}.$$

- $\forall j$, $\exists \delta_j > 0$ s.t. $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^{j+1}}$.

Observe $\{(a_j, b_j + \delta_j)\}_{j=1}^\infty$ is an open cover of

$[a+\delta, b]$ (compact!).

$\exists n > 0$ s.t. $[a+\delta, b] \subset \bigcup_{j=1}^N (a_j, b_j + \delta_j)$.

Then

$$\mu_0 (a,b] = F(b) - F(a)$$
$$< F(b) - F(a+\delta) + \frac{\varepsilon}{2}$$

$$= \mu_0 (a+\delta, b] + \frac{\varepsilon}{2}$$

$$\leq \mu_0 \left(\bigcup_1^N (a_j, b_j + \delta_j] \right) + \frac{\varepsilon}{2}$$

$$\leq \sum_1^N \mu_0 (a_j, b_j + \delta_j] + \frac{\varepsilon}{2}$$

$$= \sum_1^N (F(b_j + \delta_j) - F(a_j)) + \frac{\varepsilon}{2}$$

$$\leq \sum_1^N \left(F(b_j) + \frac{\varepsilon}{2^{j+1}} - F(a_j) \right) + \frac{\varepsilon}{2}$$

$$= \sum_1^N (F(b_j) - F(a_j)) + \sum_1^N \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2}$$

$$\leq \sum_1^N \mu_0 (a_j, b_j] + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \sum_1^\infty \mu_0 (a_j, b_j] + \varepsilon$$

Since ε was arbitrary,

$$\mu_0 (a, b] \leq \sum \mu_0 (a_j, b_j].$$

(other cases for HW.) $\square$

$$(\mathbb{R}, \mathcal{A}, \mu_0) \rightsquigarrow (\mathbb{R}, P(\mathbb{R}), \mu^*) \rightsquigarrow (\mathbb{R}, \underset{\substack{\shortparallel \\ \mathcal{M}_F}}{\mathcal{M}^*}, \underset{\substack{\shortparallel \\ \mu^*|_{\mathcal{M}_F}}}{\mu_F})$$

Observe $\mathcal{A} \subset M_F$ so $\mathcal{B}_{\mathbb{R}} \subset M_F$.

Call $\mu_F|_{\mathcal{B}_{\mathbb{R}}}$ the Lebesgue-Stieltjes measure assoc. to F.

- $F = \mathrm{id}$, call $\mu_F =: \lambda$, Lebesgue measure. $\mathcal{L} = M_{id}$.

Observe: μ_F is σ-finite.

$$\mathbb{R} = \coprod_{n \in \mathbb{Z}} (n, n+1],$$

has finite μ_F: $F(n+1) - F(n)$.

Lebesgue measure:

Dilation & translation properties:

for $E \subset \mathbb{R}$, let $rE = \{rx \mid x \in E\}$

$E + s = \{x + s \mid x \in E\}$ $\quad (r, s \in \mathbb{R})$

Thm: Suppose $E \in \mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}}$ for $\lambda|_{\mathcal{B}_{\mathbb{R}}}$.

1. For every $r \in \mathbb{R}$, $rE \in \mathcal{L}$ and $\lambda(rE) = |r| \cdot \lambda(E)$
2. For every $s \in \mathbb{R}$, $E + s \in \mathcal{L}$ and $\lambda(E+s) = \lambda(E)$

↑ uniquely characterizes λ w/ $\lambda(0,1] = 1$.

Pf we'll do ② and ① is in the notes.

Step 1: since $\mathcal{H}$ is closed under $E \mapsto E+s$,

and so is $\mathcal{B}_{\mathbb{R}}$, Then $\lambda_s(E) := \lambda(E+s)$ defines

a measure on $\mathcal{B}_{\mathbb{R}}$ s.t. $\lambda_s = \lambda|_{\mathcal{B}_{\mathbb{R}}}$.

Pf That λ_s is a measure is an exercise.

If $E \in \mathcal{H}$, then $\lambda_s(E) = \lambda(E)$.

Thus $\lambda_s = \lambda$ on $\mathcal{A}$. Thus $\lambda_s = \lambda$ on all of $\mathcal{B}_{\mathbb{R}}$

by the extension theorem.

Step 2: if $E \in \mathcal{L}$ is λ-null then $E+s$ is also λ-null.

Pf by Hw, $E \in \mathcal{L}$ is λ-null iff $\exists N \in \mathcal{B}_{\mathbb{R}}$ s.t.

$E \subset N$ and $\lambda(N) = 0$.

Now $E+s \subset N+s$ & $\lambda(N+s) = \lambda(N) = 0$ by step 1.

So $E+s \in \mathcal{L}$ is λ-null.

Now as $\mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}}$ for λ, we see $\lambda_s = \lambda$ on $\mathcal{L}$. $\square$

Facts:

① any countable set has lebesgue measure 0.

$\{x_n : n \in \mathbb{N}\} = \bigsqcup \{x_n\}$ $\quad \lambda(\{x_n\}) = 0$.

$n \in \mathbb{N}$

② The Cantor set has lebesgue measure 0

uncountable