Lec 10/14

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(X, τ) a top. sp.

A <u>Neighborhood base</u> for τ at $x \in X$ is a subset $B(x) \subset \tau$ s.t.

① $x \in V \quad \forall V \in B(x)$

② $\forall U \in \tau$ s.t. $x \in U$, $\exists V \in B$ s.t. $V \subset U$.

A <u>base</u> for τ is $\mathcal{B} \subset \tau$ that contains a nbhd base $B(x)$ $\forall x \in X$.

<u>Ex</u>: $\mathcal{B} \subset \tau$ is a base $\Leftrightarrow$ every $U \in \tau$ is a union of elts in $\mathcal{B}$.

Say (X, τ) is:

① <u>first countable</u> if $\exists$ ctble nhd base $\forall x \in X$

② <u>second countable</u> if $\exists$ ctble base

<u>Ex</u>: 2^{nd} ctble $\Rightarrow$ separable.

<u>Ex</u>: If X is 1^{st} ctble and $A \subset X$, then $x \in \bar{A}$ iff $\exists$ seq. $(x_j) \subset A$ s.t. $x_j \longrightarrow x$.

(Regularity Properties)

<u>Def</u>: A top space X is called

① T_1 if $\forall x, y$ distinct pts in X, $\exists\, \mathcal{U} \in \tau$ containing exactly one of x & y

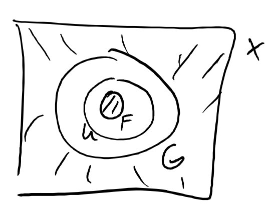
② Hausdorff (T_2) if $\forall x \neq y \in X$, $\exists$ disjoint $U, V \in \tau$ w $x \in \mathcal{U}$, $y \in V$

③ Regular (T_3) if X is T_1 and $\forall F$ closed & $x \notin F$, $\exists$ disjoint $U, V \in \tau$ s.t. $x \in \mathcal{U}$, $F \subset V$

④ Normal (T_4) if X is T_1 & $\forall$ disjoint closed $F, G \subset X$, $\exists$ disjoint $U, V \in \tau$ w/ $F \subset \mathcal{U}$, $G \subset V$.

<u>Urysohn's Lemma</u>: X normal. If $A, B \subset X$ are disjoint, nonempty, closed, then $\exists\, f : X \to [0,1]$ cts s.t. $f|_A = 0$ and $f|_B = 1$.

<u>Observe</u>: If $F \subset G \subset X$ w/ F closed & G open, $\exists$ open $\mathcal{U}$ s.t. $F \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset G$



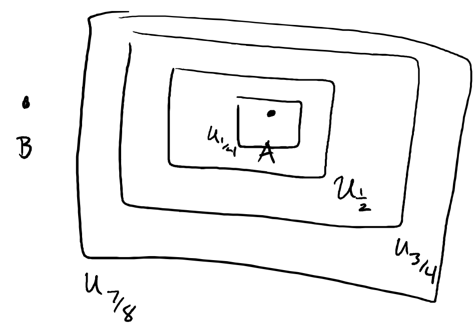
<u>Lemma²</u>: Let $D = \left\{ \frac{k}{2^n} \;\middle|\; \begin{matrix} n \in \mathbb{N}, \\ k = 1, 2, \dots, 2^n - 1 \end{matrix} \right\} \subset (0,1)$.

$\exists$ open sets $(\mathcal{U}_d)_{d \in D}$ s.t.

- $A \subset \mathcal{U}_d$ and $\overline{\mathcal{U}_d} \subset B^c \quad \forall d.$

- $\overline{\mathcal{U}}_d \subset \mathcal{U}_{d'} \quad \forall \; d < d'$

<u>Idea</u>:



et. cetera, using the observation.
(by induction)

<u>Pf of Urysohn's lemma</u>: define $f : X \longrightarrow [0,1]$

by $f(x) = \sup \{ d \mid x \notin \mathcal{U}_d \}$

Clear that $f|_A = 0$ and $f|_B = 1$.

(i) $f(x) > d \Rightarrow x \notin \overline{\mathcal{U}}_d$

$f(x) < d' \Rightarrow x \in \mathcal{U}_{d'}$

(ii) $x \notin \overline{\mathcal{U}_d} \Rightarrow f(x) \geq d$

$x \in \mathcal{U}_{d'} \Rightarrow f(x) \leq d'$

Show f is cts: Fix $x_0 \in X$ and $\varepsilon > 0$.

Case 1 Suppose $0 < f(x_0) < 1$.

Choose $d, d' \in D$ s.t. $d < f(x_0) < d'$
and $d' - d < \varepsilon$.

Then by (i) $x_0 \in U_{d'} \setminus \overline{U}_d$

By (ii), $\forall x \in U_{d'} \setminus \overline{U}_d$,

$$|f(x) - f(x_0)| < \varepsilon.$$

$\Rightarrow$ f is cts.

Case 2 $f(x_0) = 0$ or 1. Similar & omitted.

$\square$

Tietze Extension theorem: Suppose X is normal.

if $A \subset X$ is closed & $f: A \to [a,b]$ is cts
(in the relative topology on A), $\exists F: X \to [a,b]$ s.t. $F|_A = f$.

Pf wlog, $[a,b] = [0,1]$ (replace f w/ $\frac{f-a}{b-a}$).

We'll inductively build a seq of cts fns (g_n) on X s.t. $\forall n$

- $0 \le g_n \le \frac{2^{n-1}}{3^n}$

- $0 \le f - \sum_1^n g_k \le \left(\frac{2}{3}\right)^n$ on A

—

Then $\sum g_k$ converges uniformly to a cts limit fn F

and $\forall n$, $0 \leq f - F \leq f - \sum_1^n g_k \leq (\frac{2}{3})^n$ on A,

so $f = F$ on A.

Base Case: $B = f^{-1}([0, \frac{1}{3}])$, $C = f^{-1}([\frac{2}{3}, 1])$

Since f is cts, $B, C \subset A$ are closed.

Since $A \subset X$ is closed, $B, C \subset X$ are closed

By Urysohn's Lemma, $\exists$ cts $g_1 : X \to [0, \frac{1}{3}]$

s.t. $g_1|_B = 0$ and $g_1|_C = \frac{1}{3}$.

Then

$$0 \leq f - g_1 \leq \begin{cases} \frac{1}{3} - 0 = \frac{1}{3} & \text{on } B \cap A \\ \frac{2}{3} - 0 = \frac{2}{3} & \text{on } (B \cup C)^c \cap A \\ 1 - \frac{1}{3} = \frac{2}{3} & \text{on } C \cap A \end{cases}$$

$$\leq \frac{2}{3} \text{ on } A.$$

Inductive Step: If we have $g_1, \dots, g_{n-1}$,

$\exists$ cts $g_n : X \to [0, \frac{2^{n-1}}{3^n}]$ s.t.

- $g_n = 0$ on $\{f - \sum_1^{n-1} g_k \leq \frac{2^{n-1}}{3^n}\}$

- $g_n = \frac{2^{n-1}}{3^n}$ on $\{f - \sum_1^{n-1} g_k \geq (\frac{2}{3})^n\}$

$\Rightarrow f - \sum_1^n g_k \leq (\frac{2}{3})^n$ on A as before

$$\Rightarrow \quad f - \sum_1^n g_k \leq \left(\tfrac{2}{3}\right)^n \quad \text{on } A \text{ as before}$$

$$0 \leq \left(f - \sum_1^{n-1} g_k\right) - g_n \leq \left(\tfrac{2}{3}\right)^{n-1} - \frac{2^{n-1}}{3^n} = \left(\tfrac{2}{3}\right)^n. \qquad \square$$

Convergence in Top Sp:

recall $x_n \longrightarrow x$ if $\forall$ open $U \ni x$, x_n is eventually in U.
($\exists N$ s.t. $\forall n > N$)

x is a cluster point of (x_n) if $\forall$ open $U \ni x$, x_n is frequently in U
($\forall N \; \exists n > N$)

Def: A directed set is a set I equipped w/ a preorder (reflexive & transitive) not necessarily antisymmetric.
binary rel'n $\leq$ satisfying:

- $\forall\, i, j \in I$, $\exists k \in I$ s.t. $i \leq k$ & $j \leq k$.

Examples:

① $\mathbb{N}$ or $\mathbb{R}$ or any linearly ordered set

② $\mathbb{R} \setminus \{a\}$ and $x \leq y \iff |x-a| \geq |y-a|$

③ Any nbhd base for (X, τ) at $x \in X$, reverse incl. order:

$$U \subset V \iff V \leq U$$

④ X any infinite set, $\{F \subset X \mid F \text{ finite}\}$ ordered by inclusion.

idea: Sequences are fns $\mathbb{N} \longrightarrow X$

a net is a fn $I \longrightarrow X$

$\uparrow$
directed set.