Lec 9/20

Thursday, September 20, 2018 11:32

G is a semidirect product of H and N if $H \leq G$, $N \trianglelefteq G$, $H \cdot N = N \cdot H = G$, $H \cap G = \{e\}$.

$(H, N, \alpha) \rightsquigarrow N \rtimes_{\alpha} H =: G \leftarrow$ starting w/ this definition for G.

$\hookrightarrow \{(n,h) \mid n \in N, h \in H\}$

$\alpha : H \longrightarrow \mathrm{Aut}_{\mathrm{group}}(N)$

$(n_1, h_1) * (n_2, h_2) \overset{\text{def}}{=} (n_1 \cdot \alpha(h_1)(n_2), h_1 h_2)$

__Lemma__: G with operation $*$ is a group.

__Pf__ $*$ is associative (proved last time since α is a group action):

$$\alpha(h_1)(n_2) \cdot \alpha(h_1 h_2)(n_3) = \alpha(h_1)\Big(n_2 \cdot \alpha(h_2)(n_3)\Big)$$

G has identity (e_N, e_H):

$$(e_N, e_H)(n, h) = (e_N \cdot \alpha(e_H)(n), e_H h) = (n, h).$$

(n, h) has inverse $(\alpha(h^{-1})(n^{-1}), h^{-1})$ $\square$

$$H \xrightarrow{f_1} G \qquad\qquad N \xrightarrow{f_2} G$$

$$h \longmapsto (e_N, h) \qquad\qquad n \longmapsto (n, e_H)$$

We can view $H \leq G$, $N \leq G$. And $N \trianglelefteq G$

and (ii) $G = HN = NH$ and (iii) $N \cap H = \{e\}$.

$(e_N, h_1) \cdot (e_N, h_2) = (e_N \cdot \alpha(h_1)(e_N), h_1 h_2) = (e_N, h_1 h_2)$ ✓

$(n_1, e_H) \cdot (n_2, e_H) = (n_1 \cdot \alpha(e_H)(n_2), e_H e_H) = (n_1 n_2, e_H)$ ✓

So f_1, f_2 are group homomorphisms

w.t.s. $(e_N, h)(n, e_H)(e_N, h)^{-1} = (n', e_H)$ for some $n' \in N$.

$\qquad\qquad \parallel$

$(e_N \cdot \alpha(h)(n), h)(e_N, h^{-1})$

$$\overset{\shortparallel}{(\underbrace{\alpha(h)(n)}, e_H)}$$

remember, α is supposed to be like conjugation.

So $N \trianglelefteq G$, and f_1, f_2 are injective gp. hom's

So $G := N \rtimes_\alpha H$ is a semidirect product of N and H.

Now let $\mathcal{G}$ be a group, $H \leq \mathcal{G} \trianglerighteq N$; $\mathcal{G} = H \cdot N = N \cdot H$; $H \cap N = \{e\}$.

define $\alpha : H \longrightarrow \mathrm{Aut}_{gp}(N)$ by $\alpha(h)(n) = hnh^{-1}$

So $(H, N, \alpha) \rightsquigarrow G := N \rtimes_\alpha H$

Prop: $G \xrightarrow{\cong} \mathcal{G}$

$(n,h) \xmapsto{f} n \cdot h$

Pf f is a gp-hom.

$$\begin{array}{ccc} (n_1, h_1) \cdot (n_2, h_2) & = & (n_1 \cdot \alpha(h_1)(n_2), h_1 h_2) \\ \downarrow f & & \downarrow f \\ (n_1 \cdot h_1) \cdot (n_2 \cdot h_2) & \overset{\checkmark}{=} & n_1 h_1 n_2 h_1^{-1} h_1 h_2 \end{array}$$

$\mathrm{Aut}_{group}(W) = ?$

some groups, all iso's $W \xrightarrow{\cong} W$.

eg: $W = \mathbb{Z} = \langle 1 \rangle \ni 1 \longmapsto \begin{matrix} 1 \\ -1 \end{matrix}$ or

$\mathrm{Aut}_{group}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{id, \sigma\}$ $\sigma(x) = -x$.

eg $W = \mathbb{Z}/6\mathbb{Z}$. $\mathrm{Aut}_{group}(\mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{id, \sigma\}$ s.t. $\sigma(1) = 5$.

$\quad\ \ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\sigma(x) = 6 - x)$.

$\langle 1 \mid 6\cdot 1 = 0\rangle$

$1 \longmapsto 1$ or 5

eg $W = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$. Can send $1 \longrightarrow$ any $x \neq 0$. so $|\mathrm{Aut}_{gp}(\mathbb{Z}/p\mathbb{Z})| = p-1$.

$\sigma_x : 1 \longmapsto x$.

$1 \xmapsto{\sigma_x} x \xmapsto{\sigma_{xy}} xy$. $\mathrm{Aut}_{gr}(\mathbb{Z}/p\mathbb{Z}, +) \cong (\mathbb{Z}/p\mathbb{Z} \setminus \{0\}, \times)$.

exercise: Claim $\mathrm{Aut}_{gp}(\mathbb{Z}/p\mathbb{Z}, +) = (\mathbb{Z}/p\mathbb{Z}, \times) \cong (\mathbb{Z}/(p-1)\mathbb{Z}, +)$

Cor of 3rd iso. thm ($H \leq G$, $N \trianglelefteq G$, $H/H\cap N \cong H\cdot N/N$.

If $G = N \rtimes H$ then this says $H \cong G/N$).

Verbally: $G \xrightarrow{\pi} G/N$. $\forall \bar{y} \in G/N$, we can unambiguously pick

natural proj.

a representative $r_{\bar{y}} \in G$ s.t. $\mathrm{ord}_{G/N}(\bar{y}) = \mathrm{ord}_G(r_{\bar{y}})$

and $\{ r_{\bar{y}} : \bar{y} \in G/N \} = H$ is a group $\cong G/N$.

$r_{\bar{y}_1} \cdot r_{\bar{y}_2} = r_{\bar{y}_1 \bar{y}_2}$.

$\mathbb{Z}/9\mathbb{Z}$ is not a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

if it was, then it comes from some $\alpha : \mathbb{Z}/3\mathbb{Z} \xrightarrow{\text{gp hom}} \mathrm{Aut}_{gr}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

but there are no such (nonidentity) maps.

and so $\mathbb{Z}/9 = \mathbb{Z}/3 \rtimes \mathbb{Z}/3 \Longrightarrow \mathbb{Z}/9 = \mathbb{Z}/3 \times \mathbb{Z}/3$ but this is false.

Ex: $S_n \circlearrowright \mathbb{Z}^n$, $\sigma\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}$. $S_n \longrightarrow \mathrm{Aut}_{gp}(\mathbb{Z}^n)$

<u>Ex:</u> $S_n \curvearrowright \mathbb{Z}^n$, $\sigma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}$. $S_n \longrightarrow \mathrm{Aut}_{gp}(\mathbb{Z}^n)$

New group $\hat{S}_n = \mathbb{Z}^n \rtimes S_n$