MATH 5576H List of Bonus Exercises

Vilas Winstein

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1: Give an example of an infinite Abelian group which doesn't have nontrivial infinite subgroups.

2: Show that $2^{2^n} + 1$ and $2^{2^m} + 1$ are coprime for $n \neq m$.

3: Suppose $\exists c_1, c_2 > 0$ so that $c_1 < \frac{p_n}{n \log(n)} < c_2$ for all $n \geq 1$. Show that $\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$.

4: Show that the algebraic numbers are countable.

5: Show that $x \in \mathbb{Q} \Leftrightarrow x$ has a finite continued fraction expansion.

6: Find the continued fraction expansions for $\sqrt{2}$ and $\sqrt{3}$ and $\sqrt{m^2+1}$.

7: Show that the set of non-normal numbers is uncountable.

8: Show that changing the squares in Champernowne's constant to 17 gives a normal number, i.e. that the number 0.172317567817101112131415171718192021222324172 6... is normal.

9: Show that $\left[\left(\sqrt{2} - 1 \right)^n \to 0 \right] \Rightarrow \sqrt{2} \notin \mathbb{Q}$.

10: Show that $\pi(n) \sim \frac{n}{\log(n)} \Leftrightarrow p_n \sim n \log(n)$.

11: Show that the multinomial coefficients (except for 1) in $(1 + \cdots + 1)^p$ are divisible by p.

12: Show that the last nonzero base-3 digit of n^2 is 1.

13: Show that $\mathbb{Z}_p$ (the integers modulo p) is a field for prime p.

14: Show that the fields $\Gamma = \{a + bi : a, b \in \mathbb{Z}\}$ and $M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ are isomorphic.

15: Show that $\left\{ a + b\sqrt{2} : a, b \in \mathbb{Z}_5 \right\}$ is a field, and that it is isomorphic to $\left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}_5 \right\}$.

16: Show there is a finite field with p^2 elements for any prime p.

17: Show that there are infinitely many fields of characteristic p for any prime p, up to isomorphism. (The characteristic of a field is the least n so that $0 = 1 + \cdots + 1$ (n times), if such a number exists).

18: Give an example of a number-theoretical ring with infinitely many units.

19: Show that Dirichlet's theorem doesn't hold for $x \in \mathbb{Q}$. (Dirichlet's theorem states that for any $x \notin \mathbb{Q}$, there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ so that $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$).

20: Show that there are uncountably infinitely many fields of characteristic p for any prime p.

21: Prove that it is not true that if $\overline{d}(A) > 0$ then $\exists d$ such that A contains arbitrarily long arithmetic progressions with step size d (find a counterexample).

22: Show that the inequality in Hurwitz's theorem can't be improved (i.e. that there is no $c > \sqrt{5}$ so that there exist infinitely many $\frac{p}{q} \in \mathbb{Q}$ so that $\left|x - \frac{p}{q}\right| < \frac{1}{cq^2}$) for $x = \varphi$, the golden mean.

23: What are some other x for which this inequality can't be improved?

24: Are the normal numbers all transcendental?

25: If $f, g : \mathbb{N} \to \mathbb{N}$ are injective and $f \sim g$, under what conditions is $f^{-1} \sim g^{-1}$.

26: If $A = \{a_1, \dots, a_n, \dots\} \subseteq \mathbb{N}$ and $a_n \sim f(n)$ where $|f(n)| \leq P(n)$ (P is some polynomial), is it true that $AA^{-1} = \left\{\frac{a_i}{a_j} : a_i, a_j \in A\right\}$ is dense in $\mathbb{R}^+$.

27: Is it true that $\sum_{k=2}^{n} \frac{1}{k \log(k)} \sim \log(\log(n))$?

28: Is it true that $\sum_{k=1}^{n} \frac{1}{p_k} \sim \log(\log(n))$?

29: Let $G = \left\{e, a, a^2, \dots, a^{k-1}\right\}$ be a cyclic group. How many automorphisms does it have?

30: Show that $\mathbb{H}$ (the quaternions) is isomorphic to $\left\{\begin{pmatrix} u & -v \\ \bar{v} & \bar{u} \end{pmatrix} : u, v \in \mathbb{C}\right\}$.

31: Show that there is a field of 7^3 elements.

32: Create a natural rational sequence in $[0,1]$ which is uniformly distributed in $[0,1]$.

33: (Show the three statements of Szemerédi's theorem are equivalent - formulate this problem better, see september 8 notes)

34: Formulate a finitistic version of Sárközy's theorem which is equivalent to the standard version.

35: How many automorphisms does a field with p^2 elements have?

36: Show that if G is a group of prime cardinality p then G is cyclic.

37: Show that the sequence $\{n\alpha \pmod 1\}_{n=1}^{\infty}$ for $\alpha \notin \mathbb{Q}$ is uniformly distributed in $[0,1]$.

38: If $A \subseteq \mathbb{N}$, $\bar{d}(A) > 0$, is there necessarily a $d \in \mathbb{N}$ so that A contains arbitrarily long arithmetic progressions with difference d?

39: Let $A_\alpha = \left\{\left\lfloor \frac{n}{\alpha} \right\rfloor : n \in \mathbb{N}\right\}$ for $0 < \alpha \leq 1$. Prove that $d(A_\alpha) = \alpha$

40: Show that $\bar{d}(P) = 0$, where P is the set of primes.

41: Give an example of sets $A, B \subseteq \mathbb{N}$ so that $\bar{d}(A) = 1 = \bar{d}(B)$ but $A \cap B = \varnothing$.

42: Is it true that $d^*(A \cup B) \leq d^*(A) + d^*(B)$ for any $A, B \subseteq \mathbb{N}$?

43: Show that $A \subseteq \mathbb{N}$ is thick (contains arbitrarily long intervals) if and only if $d^*(A) = 1$.

44: Show that $d^*(S) < 1$ where S is the set of square-free numbers.

45: Show that $\underline{d}^*(S) = \liminf_{N\to\infty}\left(\min_m \frac{|S\cap\{m+1,\ldots,m+N\}|}{N}\right) = 0$ where S = the square-free numbers.

45: Let $a_n = \lfloor 2^c \rfloor^n$, $b_n = \lfloor 2^{cn} \rfloor$ for some $c > 1$. Prove that $d^*(\{a_n : n \in \mathbb{N}\}) = 0 = d^*(\{b_n : n \in \mathbb{N}\})$.

46: Find the formula for the nth Fibonacci number.

47: Show that if $d(A) > \frac{1}{2}$ then $x + y = z$ is soluble for $x, y, z \in A$.

48: Show that (n^c) and $\left(\log^{1+\epsilon}(n)\right)$ are dense mod 1 (for $0 < c < 1$ and $\epsilon > 0$).

49: Let $\Gamma = \{2^n 3^m : m, n \in \mathbb{N}\} = \{n_1 < n_2 < \cdots < n_k < \cdots\}$. Show that $\frac{n_{i+1}}{n_i} \to 1$.

50: Prove that x is base-r normal if and only if $\{r^n x \pmod 1 : n \in \mathbb{N}\}$ is uniformly distributed.

51: Show that $d\left(\{n_1^2 + n_2^2 : n_1, n_2 \in \mathbb{N}\}\right) = 0$.

52: Are there two density 0 sets $A, B \subseteq \mathbb{N}$ so that $AB = \{ab : a \in A, b \in B\} = \mathbb{N}$?

53: Let G be a group and $H \subseteq G$ be a subgroup. Prove that G is representable as a disjoint union of cosets of H: $G = \bigsqcup_{a\in I_l} aH = \bigsqcup_{b\in I_r} Hb$ for some $I_l, I_r \subseteq G$.

54: Show that there is no 3-dimensional complex number field.

55: Let $x, y \in \{0,1\}^{\mathbb{N}}$ be binary sequences. Show that $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$ is a metric on $\{0,1\}^{\mathbb{N}}$.

56: Show that $\{0,1\}^{\mathbb{N}}$ with the metric above is homeomorphic to C (the Cantor set).

57: Let C be the classical (middle-thirds) Cantor set. Show that $C + C = [0,2]$ and $C - C = [-1,1]$.

58: Is it true that $K + K = [0,2]$ and $K - K = [-1,1]$ where $K \subset [0,1]$ is an arbitrary Cantor set?

59: Let $\omega \in \{0,1\}^{\mathbb{N}}$ be a binary sequence. Let $\Omega = \{\sigma^k\omega : k \in \mathbb{N}\}$, where σ is a left shift of ω (truncating the initial bit: i.e. if $\omega = 10110011..$ then $\sigma\omega = 0110011...$). Show that $\overline{\Omega} = \{0,1\}^{\mathbb{N}}$ if and only if ω is weakly normal (use the metric defined on $\{0,1\}^{\mathbb{N}}$ above, $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$).

60: Let $\{x_n : n \in \mathbb{N}\} \subset [0,1]$ be a dense image of a sequence (x_n). Show that there is a permutation of the sequence (x_n), (y_n), which is u.d. mod 1.

61: x is base-2 weakly normal if every finite binary word appears in its binary expansion. Show that almost every $x \in (0,1)$ is base-2 weakly normal.

62: Show that C is not homeomorphic to $[0,1]$.

63: Show that $[0,1]$ does not have measure zero.

64: Let (a_n) be a sequence so that $0 < \sum_{n=1}^{\infty} a_n < 1$. Construct a set by removing the open interval of length a_1 centered at $\frac{1}{2}$ from $[0,1]$, then removing two open intervals each of length $\frac{a_2}{2}$ from the left and right intervals of the remaining set, with each removed interval centered at the midpoint of the interval of the remaining set. Continue by removing an open interval of length $\frac{a_3}{4}$ from each of the four remaining contiguous intervals (again centered at the middle). Let $A = \bigcap_{i=1}^{\infty} K_i$, where K_i is the set obtained by stopping after i iterations of the above process. Show that A does not have measure 0.

65: Show that the sets $\{0,1\}^{\mathbb{N}}, \{0,1\}^{\mathbb{Z}}, \{0,1,\dots,m-1\}^{\mathbb{N}}$, and $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ are all homeomorphic.

66: Let $(x_n) \in \{0,1\}^{\mathbb{N}}$ be the binary expansion of a base-2 normal number x. Show that the number whose binary expansion is (x_{an+b}) is normal.

67: Let a set B be constructed by removing an arbitrary number of open intervals from $[0,1]$, then removing some other arbitrary number of open intervals from the remaining set, and continuing this process ad infinitum only requiring that the intervals do not touch and that the total length of the (closed) intervals in the remaining set decreases after each step to a limiting value of 0. Show that B is homeomorphic to C (the Cantor set)

68: Let $S \subset P$ and let $A_S = \{x : x = \prod_{p \in S} p^{\alpha_p}, \alpha_p \in \mathbb{N}\}$. Is it true that $d(A_S) > 0$ if $\sum_{p \in P \setminus S} \frac{1}{p} < \infty$?

69: Let $P_1 \sqcup P_2 = P$ so that $\sum_{p \in P_1} \frac{1}{p} = \sum_{p \in P_2} \frac{1}{p} = \infty$. And let $A_i = \{x : x = \prod_{p \in P_i} p^{\alpha_p}, \alpha_p \in \mathbb{N}\}$ for $i = 1,2$. Is it true that $d(A_1) = d(A_2) = 0$?

70: Show that D_f, the set of points of discontinuity of a monotone function f, is at most countable.

71: Show that the endpoints of the removed intervals in the iteration process of constructing C, the classical Cantor set, are dense in C.

72: Show that $\ell_\infty(\mathbb{N})$, the set of all bounded sequences in $\mathbb{R}^{\mathbb{N}}$, with metric $d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$, is a metric space.

73: Consider the sequence $1,2,4,8,1,3,6,1,\dots$, the first digit of 2^n. Show that the symbol 7 is more frequent than the symbol 8 in this sequence (i.e. the density of the indices where 7 appears is greater than the density of the indices where 8 appears).

74: Show that $C[0,1]$ with the metric $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$ is complete metric space.

75: Let $B = \{f \in C[0,1] : \|f\| := \max_{x \in [0,1]} |f(x)| \leq 1\}$ is bounded and closed but is not compact.

76: Show that $C + C = [0,2]$ by using the fact that $C + C$ is dense in $[0,2]$.

77: Show that $\{0,1\}^{\mathbb{N}}$ with the metric $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$ is a compact metric space.

78: Show that $\mathbb{R}[x]$ being dense in $C[0,1]$ is equivalent to trigonometric polynomials (finite linear combinations of $e^{2\pi i n x}$ for varying n) being dense in $C[0,1]$ with metric $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$.

79: Is there a Cantor set containing only irrational numbers?

80: Show that $n^2\alpha$ is u.d. mod 1.

81: Show that $((n\alpha, n\beta)) \in \left([0,1]^2\right)^{\mathbb{N}}$ is u.d. mod 1 iff α and β are linearly independent over $\mathbb{Q}$.

82: Let $f(x) \in \mathbb{R}[x]$ and assume that at least one coefficient of f (other than the constant term) is irrational. Show that $(f(n))$ is u.d. mod 1.

83: Show that for distinct primes p and q, $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

84: Generalize a proof given in class about the irrationality of $\sqrt{2}$ to any quadratic irrationality.

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86: Give an analogue in Gaussian integers of Furstenburg's proof of the infinitude of primes.

87: Show that $d^*(P) = 0$.

88: Show that if $\frac{a_{n+1}}{a_n} \geq \lambda > 2$ for all n, then $d(FS(a_n)) = 0$ where $FS(a_n) = \left\{\sum_{i=1}^{k} a_{n_i} : n_i \in \mathbb{N}\right\}$.

89: Show, using Schur's Theorem, that for any $n \in \mathbb{N}$, if $p \in P$ is large enough then there are distinct x, y, and z, not congruent to 0 mod p, so that $x^n + y^n \equiv z^n \pmod{p}$.

90: Show that "$\mathbb{N} = \bigcup_{i=1}^{r} C_i \Rightarrow x, y, x+y \in C_i$ for some x, y and i" is equivalent to Schur's Theorem.

91: Formulate a finitistic version of Van Der Warden's Theorem and prove its equivalence to the standard version.

92: Show that $x + y = 3z$ is not partition regular.

93: Show that the set of square-free numbers is not syndetic.

94: Give countably many counterexamples to problem 8 on the midterm (which states that for any $x \notin \mathbb{Q}$ with $x > 1$, $\{x^n : n \in \mathbb{N}\}$ is dense in $[0,1]$) using the idea discussed in class.

95: Show that $\mathbf{1}_C$, the characteristic function of the Cantor set, is a pointwise-converging limit of a sequence of continuous functions on $[0,1]$.

96: Show that $\left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\right\}$ has multiplicative inverses for all nonzero elements.

97: Show that $\left\{a_0 + a_1\sqrt[5]{2} + a_2\sqrt[5]{2^2} + a_3\sqrt[5]{2^3} + a_4\sqrt[5]{2^4} : a_i \in \mathbb{Q}\right\} \setminus \{0\}$ has multiplicative inverses.

98: Show that $\left\{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q}\right\} \setminus \{0\}$ has multiplicative inverses.

99: Show that if $d^*(A) > 0$ then $A - A$ is syndetic.

100: Prove that $\{n : n\alpha \bmod 1 \in (a,b)\}$ and $\{n : n^2\alpha \bmod 1 \in (a,b)\}$ are syndetic.

101: Show that the sequence $(2^n x)$, for $n \in \mathbb{N}$, is not well-distributed mod 1 for any x.

102: Show that Champernowne's number $0.1234567891011121314\ldots$ is transcendental.

103: Show that $d^*(S) = 0$ where $S = \{a^2 + b^2 : a, b \in \mathbb{N}\}$.

104: Show that the set of square-free numbers is not piecewise syndetic.

105: Show that for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, at least one C_i is piecewise syndetic.

106: Show that if S is piecewise syndetic and $S = \bigcup_{i=1}^{r} C_i$ then at least one C_i is piecewise syndetic.

107: Show that if f_1 and f_2 are Bohr almost periodic (B.A.P.) then $f_1 + f_2$ and $f_1 \cdot f_2$ are too.

108: Show that $\sin(n)$ is B.A.P.

109: Show that $\mathbf{1}_A$, the characteristic function of $A = \{\lfloor n\alpha \rfloor, n \in \mathbb{Z}\}$ is B.A.P.

110: Show that $\sin(x) + \cos(\sqrt{2}x)$ is B.A.P. on $\mathbb{R}$ (give a good definition for B.A.P. on $\mathbb{R}$).

111: Show that $\mathbf{1}_{\mathbb{Q}}$, the characteristic function of $\mathbb{Q}$ cannot be a pointwise-converging limit of a sequence of continuous functions on $[0,1]$.

112: Show, using Van der Warden's theorem, that piecewise syndetic sets are AP-rich.

113: Show that $P - 1$ and $P + 1$ are divisible sets.

114: Show that $P \pm d$ is not divisible if $d \neq 1$.

115: A finitistic version of Sárközy's Theorem is: For any $\epsilon > 0$ there is an $L \in \mathbb{N}$ so that if $b - a > L$ then for any $A \subseteq \{a, a + a, \ldots, b\}$ with $\frac{|A|}{b-a} > \epsilon$ there is an $n \in \mathbb{N}$ and some $x, y \in \mathbb{A}$ so that $x - y = n^2$. Prove this statement's equivalence with the standard form of Sárközy's Theorem.

116: Show that $\bar{d}\big(A \cap (A - n^2)\big) > 0$ if and only if there is an n so that $A \cap (A - n^2) \neq \varnothing$.

117: Show that $\varphi_A(n) = \bar{d}\big(A \cap (A - n^2)\big)$ is positive definite for any $A \subseteq \mathbb{N}$ with $\bar{d}(A) > 0$.

118: Show that $\varphi(n) = \langle U^n v, u \rangle$ is positive definite for any unitary matrix $U \in M_n(\mathbb{C})$ and $v, u \in \mathbb{C}^m$.

119: Show that $\varphi(n) = \int_{\mathbb{T}} e^{2\pi i n x}\, d\mu$ is positive definite.

120: Show that $q_{r+1} > q_{r-1}$ if q_n is the denominator of the nth convergent to a continued fraction.

121: As in the above exercise, find out what the next number is after $\sqrt{5}$ and $\sqrt{8}$ (if we don't allow a_r to be 2 this time).

122: Show that if $\bar{d}(A), \bar{d}(B) > 0$ then $(A - A) \cap (B - B)$ is syndetic.

123: Factor $x^3 + Dy^3 + D^2 z^3 - 3Dxyz = 1$ if D is a cube. Thus show the equation is insoluble.

124: Give a matrix representation of the Pellians corresponding to $\mathbb{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathbb{Z}[\sqrt[4]{2}]$.

125: Show that if $f : (0,1) \to \mathbb{R}$ is continuous and satisfies $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$ for any $x_1, x_2 \in (0,1)$ (i.e. f is midpoint convex) then $f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ for any $\alpha_1, \alpha_2 \geq 0$ where $\alpha_1 + \alpha_2 = 1$ and for any $x_1, x_2 \in (0,1)$ (i.e. f is convex in the standard sense).

126: Give an example of a discontinuous function on $(0,1)$ which is midpoint convex but not convex.

127: Show that $\alpha_1 x_1 + \alpha_2 x_2 \geq x_1^{\alpha_1} x_2^{\alpha_2}$ for any $\alpha_1, \alpha_2 \geq 0$ where $\alpha_1 + \alpha_2 = 1$.

128: Show that $\alpha_1 x_1 + \cdots + \alpha_k x_k \geq x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ for any $\alpha_1, \ldots, \alpha_k \geq 0$ where $\alpha_1 + \cdots + \alpha_k = 1$.

129: Show that if f is convex then $f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \leq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$ for any $\alpha_1, \ldots, \alpha_k \geq 0$ where $\alpha_1 + \cdots + \alpha_k = 1$.

130: Suppose $d(A), d(B)$ exist. Is it necessarily true that $d(A \cap B)$ or $d(A \cup B)$ exist?

131: Show that f is continuous (by the ϵ-δ definition) iff for any open set U, $f^{-1}(U)$ is open.

132: Show that an additive subgroup S of $\mathbb{R}^n$ is a lattice if and only if S is discrete.

133: Show that if $d^*(A) > 0$ then for any increasing sequence $(n_i) \subseteq \mathbb{N}$ there exist indices $i < j$ so that we have $d^*(A \cap (A - (n_j - n_i))) > 0$.

134: Show that $\{\lfloor n\alpha \rfloor : n \in \mathbb{N}\}$ is a set of recurrence for any irrational α.

135: Show that any thick set contains a set of differences.

136: Let G_i be the set of #s representable as a product of i distinct prime powers. Show $d(G_i) = 0$.

137: Show that if $\overline{d}(A) = 1$ then $\overline{d}(A \cap (A - n)) = 1$ for any $n \in \mathbb{N}$.

138: Show that if $d^*(A) = 1$ then $d^*(A \cap (A - n)) = 1$ for any $n \in \mathbb{N}$.

139: Let $f : \mathbb{Z} \to \mathbb{C}$ be a bounded function such that there is a sequence (N_i) which increases to ∞ and $\lim_{i\to\infty} \frac{1}{N_i} \sum_{k=1}^{N_i} f(k)\overline{f(k+n)}$ exists. Show that $\varphi(n) = \lim_{i\to\infty} \frac{1}{N_i} \sum_{k=1}^{N_i} f(k)\overline{f(k+n)}$ is positive definite.

140: Show that $\varphi(n) = \langle U^n f, f\rangle$ is positive definite for any f if U is unitary.

141: Show that $\rho(A,B) = \mu(A \triangle B)$ is a pseudometric on the set of measurable sets in a measure space (namely, show that the triangle inequality holds).

142: Show that the transformation T given by $x \mapsto 2x \pmod 1$ on $[0,1]$ is ergodic.

143: Show that the transformation T given by $x \mapsto x + \alpha \pmod 1$ on $[0,1)$ is ergodic iff $\alpha \notin \mathbb{Q}$.

143: Show that for any unitary linear transformation $U : \mathbb{C}^n \to \mathbb{C}^n$, $\mathbb{C}^n$ can be decomposed into the direct sum $\mathbb{C}^n = \{x : Ux = x\} \oplus \operatorname{Span}\{x - Ux : x \in \mathbb{C}^n\}$.

144: Derive from the fact that $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to \int f$ for almost every x and for every f iff T is ergodic and the fact that $x \mapsto 2x \pmod 1$ is ergodic that almost every number in $[0,1)$ is base-2 normal.

145: Show that $\frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ iff $\frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^{-n}A) \to \mu^2(A)$.

146: Modify the proof that in any 3 consecutive convergents to a continued fraction for an irrational number w there is at least one $\frac{p_n}{q_n}$ with $\left|\frac{p_n}{q_n} - w\right| < \frac{1}{\sqrt{5}q_n^2}$. Show instead that if we don't allow a_r to be 1 we can replace $\sqrt{5}$ by $\sqrt{8}$ in the resulting statement.

147: Using the integral $\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} = \frac{\pi^2}{6}$, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

148: Show that if $S \subset \mathbb{R}$ and $\mu(S) > 1$ then $S - S$ contains a nonzero integer.

149: Let L be a lattice in $\mathbb{R}^2$ with fundamental domain $\Delta_L = \{\alpha_1 x_1 + \alpha_2 x_2 : 0 \le \alpha_1, \alpha_2 < 1\}$ (where x_1, x_2 are the basis vectors of L). Show that if $S \subset \mathbb{R}^2$ with $\mu(S) > \mu(\Delta_L)$ then $S - S$ contains a nonzero point of L.

150: Let L be a lattice in $\mathbb{R}$ as above. Show that if $K \subset \mathbb{R}^2$ is a symmetric convex body with $\mu(K) > 4\mu(\Delta_L)$ then K contains a nonzero point of L.

151: Show that for any prime p, the multiplicative group $\mathbb{Z}_p^*$ of invertible elements of $\mathbb{Z}/p\mathbb{Z}$ is cyclic.

152: Let p be a prime of the form $4k+1$, and let $u \in \{1, \dots, p-1\}$ so that $u^2 \equiv -1 \pmod p$. Now Let $L = \left\{(x,y) \in \mathbb{Z}^2 : y \equiv ux \pmod p\right\}$. Show that L is a lattice with $\mu(\Delta_L) = p$, and use this result to deduce that $p = a^2 + b^2$ for some integers a, b.

153: Show, using an ergodicity argument, that if $A \subseteq \mathbb{N}$ with $d(A) > 0$ then $A - A$ is syndetic.

154: Show that the countable union of sets which are closed and nowhere dense cannot cover $[0,1]$.

155: Show that the set $\frac{P}{P} = \left\{\frac{p}{q} : p, q \in P\right\}$ is dense in $\mathbb{R}_+$.

156: Let $G = \{g_1, \dots, g_n\}$ be a group, and let its group algebra over $\mathbb{R}$ be $\{\sum_{i=1}^n \alpha_i g_i : \alpha_i \in \mathbb{R}\}$. If multiplying two elements of this set gives $\left(\sum_{i=1}^n \alpha_i g_i\right)\left(\sum_{i=1}^n \beta_i g_i\right) = \sum_{i,j=1}^n \alpha_i \beta_j g_i g_j = \sum_{k=1}^n \gamma_k g_k$, give a formula for γ_k.

157: Show that the set of Liouville numbers is dense G_δ.

158: Show that Champernowne's number 0.123456789101112... is transcendental but not Liouville.

159: Show that the set of Liouville numbers contains a shift of any countable set.

160: Give an example of $A \subseteq \mathbb{N}$ where $d^*(A) > 0$ but $A + A$ is not syndetic.

161: Show that the numbers $\left\{\sqrt{p_n} : n \in \mathbb{N}\right\}$ are linearly independent over $\mathbb{Q}$.

162: Prove that any metric space X can be completed (similarly to the proof in Katok).

163: Let $h(x) = \begin{cases} 0 & \text{if } x \notin [-1,1] \\ 1 & \text{if } x \in [-1,1] \end{cases}$. For any $\epsilon > 0$, construct $f \in C^\infty$ so that $\int_{\mathbb{R}} |f(x) - h(x)| dx < \epsilon$

164: Let $P_k = \{2^{e_1} 3^{e_2} 5^{e_3} \cdots p_k^{e_k} : e_i \in \mathbb{N}\}$. Show that $d^*(P_k) = 0$.

165: Prove that $f \in C[0,1]$ is uniformly continuous.

166: Let $\ell^2_{\mathbb{R}} = \left\{(x_n) : x_n \in \mathbb{R}, \sum_{i=1}^\infty x_i^2 < \infty\right\}$, and $d(x,y) = \sqrt{\sum_{i=1}^\infty (x_i - y_i)^2}$ for $x = (x_n), y = (y_n) \in \ell^2_{\mathbb{R}}$. Show that $\left(\ell^2_{\mathbb{R}}, d\right)$ is a metric space.

167: Show that $\left(\ell^2_{\mathbb{R}}, d\right)$ as defined above is a complete metric space.

168: In a finite field $\mathbb{Z}/p\mathbb{Z}$, what is the number e?

169: What can you say about quadratic equations over $\mathbb{H}$?

170: Show that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

171: Find a value of $\alpha \notin \mathbb{Q}$ for which the set $\{n^2\alpha - m^2 : n, m \in \mathbb{Z}\}$ is not dense in $\mathbb{R}$.

172: Find all values of $\alpha \notin \mathbb{Q}$ for which the set $\{n^2\alpha - m^2 : n, m \in \mathbb{Z}\}$ is not dense in $\mathbb{R}$.

173: Show that for any sequence $(n_i) \subseteq \mathbb{N}$ with $n_i \to \infty$ as $i \to \infty$, the set of finite sums denoted by $FS\left((n_i)\right) = \{n_{i_1} + \cdots + n_{i_k} : i_1 < \cdots < i_k, k \in \mathbb{N}\}$ has $\left(FS\left((n_i)\right)\right) \cap m\mathbb{N} \neq \varnothing$ for all $m \in \mathbb{N}$.

174: Is it true that any $A \subseteq \mathbb{N}$ with $d(A) > 0$ contains a shift of $FS((n_i))$ for some $n_i \nearrow \infty$?

175: Call $A \subseteq \mathbb{N}$ an IP* set if $A \cap FS((n_i)) \neq \varnothing$ for all $n_i \nearrow \infty$. Show that if A_1, A_2 are IP*, then $A_1 \cap A_2$ is also IP*.

176: Verify that if $n^2\alpha$ mod 1 gets arbitrarily close to 0 or 1 then $n^2\alpha$ mod 1 is dense in $[0,1]$.

177: Prove that $n^3\alpha$ mod 1 is dense in $[0,1]$ using similar reasoning as in class.

178: Prove that if $\alpha \notin \mathbb{Q}$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\mathbb{N} = \{\lfloor n\alpha \rfloor : n \in \mathbb{N}\} \sqcup \{\lfloor n\beta \rfloor : n \in \mathbb{N}\}$.