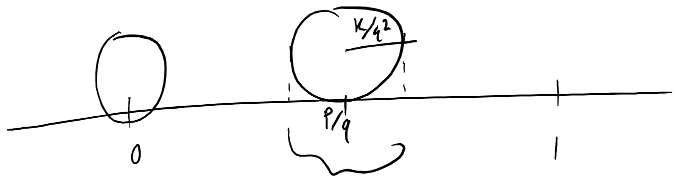
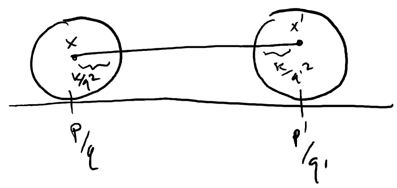
Caleb Talk

Monday, November 20, 2017 14:13

The geometry of Ford Circles



well approximable by p/q.



want tangency: $d(x,x') = \frac{k}{q^2} + \frac{k}{q'^2} =$

$$\| \quad \sqrt{\left(\frac{p}{q}-\frac{p'}{q'}\right)^2 + \left(\frac{k}{q^2}-\frac{k}{q'^2}\right)^2}$$

$$\Longrightarrow \quad (p'q - pq')^2 = 4k^2$$

$$\vee\!/ \quad 1$$

if $k = \frac{1}{2}$, circles are tangent iff $\begin{vmatrix} p & q \\ p' & q' \end{vmatrix} = \pm 1$.

and are external o.w.

So circles tesselate.

Def: $C_{p/q}$ is ford circle on p/q.

$C_{p/q}$, $C_{p'/q'}$ tangent iff $\begin{vmatrix} p & p' \\ q & q' \end{vmatrix} = \pm 1$, o.w. are, Proof: $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$ are tangent iff $\begin{vmatrix} p & p' \\ q & q' \end{vmatrix} = \pm 1$, o.w. are wholly external.

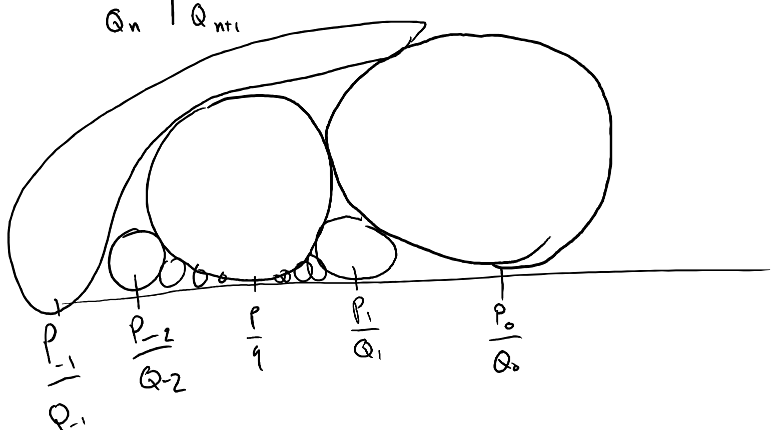
Def: $\frac{p}{q}$, $\frac{p'}{q'}$ are adjacent $\left(\frac{p}{q} \,\middle|\, \frac{p'}{q'}\right)$ if $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$ are tangent.

Theorem: If $\frac{p}{q} \,\Big|\, \frac{P_0}{Q_0}$ then all the other fractions adj to $\frac{p}{q}$ are

$$\frac{P_n}{Q_n} = \frac{P_0 + np}{Q_0 + nq}$$

Proof $\frac{P_n}{Q_n} \,\Big|\, \frac{p}{q}$ since $P_n q - Q_n p = P_0 q + nqp - Q_0 p - nqp = \pm 1$

$\frac{P_n}{Q_n} \,\Big|\, \frac{P_{n+1}}{Q_{n+1}}$ since $(P_0 + np)(Q_0 + (n+1)q) - (Q_0 + nq)(P_0 + (n+1)p) = \pm 1$



$\frac{P_n}{Q_n} - \frac{p}{q}$ $\begin{array}{l} n \to \infty \\ \text{vs} \\ n \to -\infty \end{array}$ to 0 from right v left

no more room to fit more circles

if $\frac{p}{q} \,\Big|\, \frac{p'}{q'}$ then the mediant $\frac{p''}{q''}$ is the unique

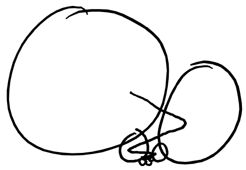
$\frac{p}{q} \,\Big|\, \frac{p''}{q''} \,\Big|\, \frac{p'}{q'}$. $\quad \frac{p''}{q''} = \frac{p+p'}{q+q'}$

$\frac{p_1}{q_1} = \frac{0}{1}$, $\frac{P_0}{Q_0} = \frac{1}{0}$. All adjacencies to $\frac{p_1}{q_1}$ are $\frac{1}{n}$

adjacent

given $[0; a_1, \ldots]$, then $\frac{p_2}{q_2} = \frac{1}{a_1}$, $\frac{p_3}{q_3} = \frac{p_1 + n p_2}{q_1 + n q_2}$.

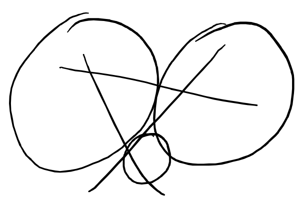
Let $n = a_2 \Rightarrow \frac{p_3}{q_3} = \frac{a_2 p_2 + p_1}{a_2 q_2 + q_1}$, ...



count cw or ccw, go around a_n times

depending on parity of n.

<u>Theorem</u> If L is a path above the real line, and L intersects (in succession) $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$, then $\frac{p}{q} \,\Big|\, \frac{p'}{q'}$.



<u>Def</u> A farey sequence F_n is the collection $\left\{ \frac{p}{q} \in \mathbb{Q},\ q \le n \right\}$

take L to be a horizontal line $y = \alpha$.

the circles intersected are a farey sequence.

<u>Cor</u>. Consec. terms in F_n are adjacent.

<u>Cor</u> anything added from F_n to F_{n+1} is a mediant of 2 consec. elements of F_n.

if L is $y = mx$ and L intersects $r_1, \dots, r_k$

then $\{r_1, \dots, r_k\} = \{\frac{p}{q} \in \mathbb{Q} : pq \le n\}$, for some $n(m)$.

$$\begin{array}{c} \| \\ F'_n \end{array}$$ ("generalized" farey)

<u>Theorem</u>: if $\alpha \notin \mathbb{Q}$ then $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ has infinitely many solutions.

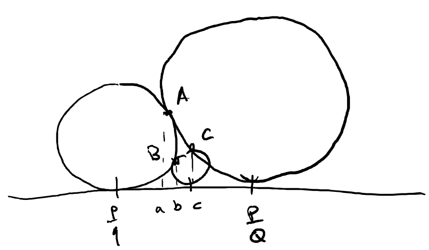
take L to be $x = \alpha$.

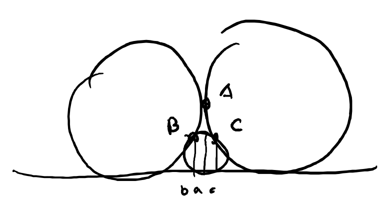
<u>Thm</u>: if $\overset{=\{x=\alpha\}}{L}$ crosses a mesh triangle, then one of the rationals whose circle bounds the triangle satisfies

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

<u>Proof</u>: Let $\frac{P}{Q}$, $\frac{p}{q}$, $\frac{p_1}{q_1}$ bound a mesh triangle that L intersects. Say $Q \le q$, $p_1 = p + P$, $q_1 = q + Q$,

and $\frac{P}{Q} > \frac{p}{q}$ (can get here by reflecting), so that $Pq - Qp = 1$





Solution is: $\frac{P}{Q}$
(well-approxing rational)

Solution is: $\frac{p_1}{q_1}$

$\mid \quad \frac{p}{} \quad \mid \quad P$

$$a = \frac{\frac{1}{2Q^2}\frac{p}{q} + \frac{1}{2q^2}\frac{P}{Q}}{\frac{1}{2Q^2} + \frac{1}{2q^2}}$$

$$= \frac{pq + PQ}{q^2 + Q^2}$$

$$b = \frac{pq + p_1 q_1}{q^2 + q_1^2}$$

$$c = \frac{p_1 q_1 + PQ}{q_1^2 + Q^2}$$

$$b - a = \frac{q^2 - qQ - Q^2}{(q^2 + Q^2)(q^2 + q_1^2)}$$

Let $S = \frac{q}{Q}$ (≥ 1). Then $S^2 - S - 1$ has same sign as $b-a$.

<u>Case 1</u> $a < b$, so $S > \frac{\sqrt{5}-1}{2}$

$$\left|\frac{P}{Q} - \alpha\right| = \frac{P}{Q} - \alpha \leq \frac{P}{Q} - a$$

$$= \frac{P}{Q} - \frac{pq + PQ}{q^2 + Q^2}$$

$$= \frac{q}{Q(q^2 + Q^2)} = \frac{S}{S^2+1}\,\frac{1}{Q^2}$$

$$< \frac{1}{\sqrt{5}\,Q^2}$$

Case 2 $b < a$, so $S < \frac{1}{2}(\sqrt{5}+1)$

Case 2 $b < a$, so $s < \frac{1}{2}(\sqrt{5}+1)$

$$\left|\frac{p_1}{q_1} - \alpha\right| \le \frac{p_1}{q_1} - b \qquad \text{since } c \text{ is closer to } \frac{p_1}{q_1} \text{ since it's higher on the circle.}$$

$$= \frac{s(s+1)}{s^2+(s+1)^2}\,\frac{1}{q_1^2}$$

$$< \frac{1}{\sqrt{5}\,q_1^2}$$

∎

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1,\ a,b,c,d \in \mathbb{Z} \right\}.$$

Def $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \dfrac{az+b}{cz+d}$

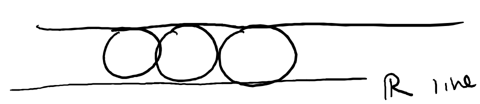
Then $\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) z = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} z\right)$.

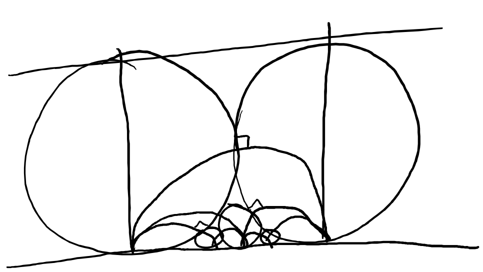
Prop $SL_2(\mathbb{Z})$ acts on ford circles

In particular, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} C_{\frac{p}{q}} = C_{\frac{ap+bq}{cp+dq}}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, but $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C_{\frac{0}{1}} = C_{\frac{1}{0}}$

so $C_{\frac{1}{0}} =$





"The" Farey Diagram

Prop if we let $S_{\frac{p}{q},\frac{p'}{q'}}$ be semicircle

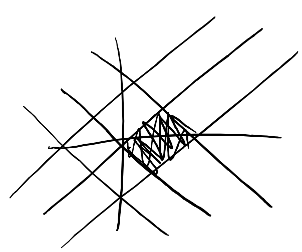
joining $\frac{p}{q}$ | $\frac{p'}{q'}$ at right angles,

$S_{\frac{p}{q},\frac{p'}{q'}}$ goes through $C_{\frac{p}{q}} \cap C_{\frac{p'}{q'}}$

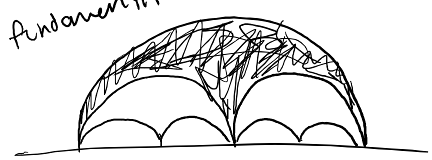
all of these semicircles are straight lines in hyperbolic geometry.

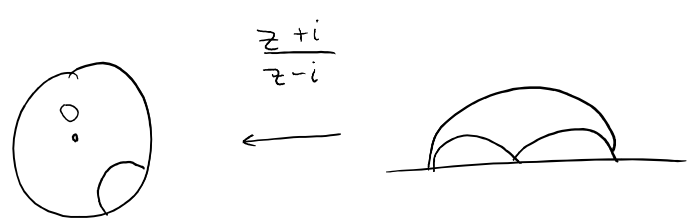
$SL_2(\mathbb{Z})$ actions preserve these lines.

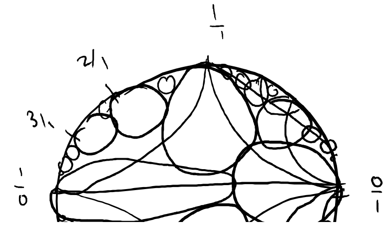
The Farey Diagram is a "Lattice"

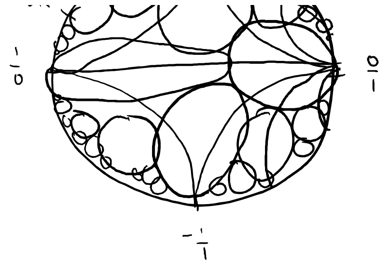


fundamental domain, convex







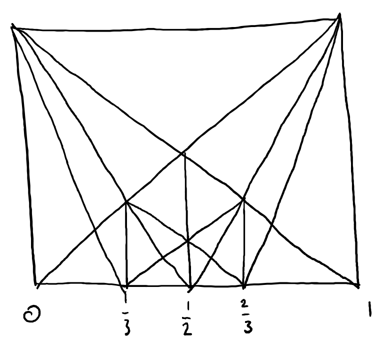


Exercises:

1: Find the total area of the ford circles on the unit interval $[0,1]$. (use Ch 17 from Hardy).

Freestyle: Ford spheres w/ Complex Rationals?

2: Prove this construction gives farey sequences (or fix it + prove)



3: Use the cayley transform $\frac{z+i}{z-i}$ to parametrize pythagorean triples.

(it takes $\mathbb{R}$ to boundary of circle)