Lec 2/26

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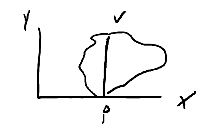

Lemma Let X and Y be top. sp. with Y compact. Let $p \in X$ and suppose that V is open in $X \times Y$ and $\{p\} \times Y \subseteq V$. Then there is a set U open in X s.t. $p \in U$ and $U \times Y \subseteq V$.

Pf Let $\mathcal{S} = \{(A,B) : A \text{ open in } X, B \text{ open in } Y, A \times B \subseteq V\}$. Then, $\bigcup_{(A,B) \in \mathcal{S}} A \times B = V$.

Since V is open in $X \times Y$. $V \supseteq \{p\} \times Y$ which is compact since Y is compact.

thus $\exists$ a finite subcollection of $\mathcal{S}$ which covers $\{p\} \times Y$.

Hence $\exists\, n \in \mathbb{N}$, $\exists\, (A_1, B_1), \dots, (A_n, B_n) \in \mathcal{S}$, $\{p\} \times Y \subseteq \bigcup_{k=1}^{n} A_k \times B_k$. We may diskard values of k for which $p \notin A_k$, hence wolog $p \in A_k \; \forall k$.

Let $U = \bigcap_{k=1}^{n} A_k$. Then $p \in U$ and U is open in X. Let $(x,y) \in U \times Y$.

then $x \in U$ and $y \in Y$. $y \in B_k$ for some k, so $(x,y) \in U \times B_k \subseteq A_k \times B_k \subseteq V$.

thus $U \times Y \subseteq V$. $\square$

Propn Let X and Y be top sp with Y compact. Let $f : X \times Y \to \mathbb{C}^\times$ be continuous. Let $p \in X$. Suppose $y \mapsto f(p,y)$ has a continuous logarithm $y \mapsto \varphi_p(y)$. (a) Then $\exists (U, \varphi_U)$ s.t. U is an open nhd of p in X, φ_U is a continuous logarithm of $f|_{U \times Y}$ and $\varphi_U(p,y) = \varphi_p(y)$ $\forall y \in Y$.

(b) Suppose in addition that Y is connected. Then for any two such pairs (U, φ_U) and (V, φ_V), there is an open nhd W of p in X s.t. $W \subseteq U \cap V$ and for each $(x,y) \in W \times Y$, $\varphi_U(x,y) = \varphi_V(x,y)$.

Pf Case 1: Suppose $f(p,y) = 1 \; \forall y \in Y$ and $\varphi_p(y) = 0$. Let $W = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

W is open and does not contain the origin. Let $V = f^{-1}(W)$

then V is open in $X \times Y$ and $\forall y \in Y$, $f(p,y) = 1 \in W$ so

$(p, y) \in V$. Thus $\{p\} \times Y \subseteq V$. Hence, by the lemma, $\exists\, U$ an open nhd of p in X s.t. $U \times Y \subseteq V$. define φ_U on $U \times Y$ by $\varphi_U(x, y) = \mathrm{Log}\,(f(x, y))$ (principal Log).

Then φ_U is a continuous log.

Case 2: (general) Define $g : X \times Y \longrightarrow \mathbb{C}^\times$ by $g(x, y) = \dfrac{f(x, y)}{f(p, y)}$.

then g is continuous and $\forall y \in Y$, $g(p, y) = 1$.

hence case 1 applies to g, so $\exists\, (U, \psi_U)$ s.t. U is an open nhd of p in X and ψ_U is a continuous logarithm of $g|_{U \times Y}$. And $\forall y \in Y$, $\psi_U(p, y) = 0$.

define $\varphi_U : X \times Y \longrightarrow \mathbb{C}^\times$ by $\varphi_U(x, y) = \psi_U(x, y) + \varphi_p(y)$.

then $e^{\varphi_U(x, y)} = g(x, y)\, f(p, y) = f(x, y)$.

and $\varphi_U(p, y) = \psi_U(p, y) + \varphi_p(y) = \varphi_p(y)$ so (a) is proved.

b) Suppose Y is connected, and (U, φ_U), (V, φ_V) are two such pairs. Then φ_U and φ_V are cts logs on $(U \cap V) \times Y$. define K on $(U \cap V) \times Y$ by $K(x, y) = \dfrac{\varphi_U(x, y) - \varphi_V(x, y)}{2\pi i}$. K is locally constant as a continuous integral fn.

$K(p, y) = 0$, so $\exists$ nhd of (p, y) where K is 0.

Fix $y_0 \in Y$ (WLOG, $Y \neq \emptyset$)

Union all nhds to get a nhd of $\{p\} \times Y$ where K is 0.

Theorem Let X and Y be top. sp. with Y compact and connected.

Let $q \in Y$ and suppose the fn $x \longmapsto f(x, q)$ has a cts log, say φ^q. Suppose also that $\forall x \in X$, the fn $y \longmapsto f(x, y)$ has a cts log. Then f has a unique continuous logarithm φ in $X \times Y$ such that $\forall x \in X$,

$\varphi(x, q) = \varphi^q(x)$.

Pf For each $x \in X$, the fn $y \longmapsto f(x,y)$ has a unique cts logarithm φ_x satisfying $\varphi_x(q) = \varphi^q(x)$.

Define $\varphi : X \times Y \longrightarrow \mathbb{C}$ by $\varphi(x, y) =$