Lec 2/12

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Lemma. Suppose Y is a topological space, let $A \subseteq X \subseteq Y$. Let $B = Cl_X(A)$, $C = Cl_Y(A)$. Then $B = C \cap X$

Proof: $C \cap X$ is closed in X since C is closed in Y. $A = A \cap X \subseteq C \cap X$. Thus $C \cap X$ is closed in X and contains A. But B is the smallest closed set in X containing A so $B \subseteq C \cap X$. B is closed in X and so $\exists\, D \subseteq Y$ s.t. $B = D \cap X$. $A \subseteq B \subseteq D$, so D is a closed subset of Y containing A so $D \supseteq C$. Thus $D \cap X \supseteq C \cap X$ but then $B \supseteq C \cap X$ so $B = C \cap X$. $\square$

Thm Let X be a top. sp. let C be a connected subset of X. Let $C \subseteq E \subseteq \overline{C}$ (in X). Then E is connected.

Pf Let U be clopen in E. Then $U \cap C$ is clopen in C, so since C is connected, $U \cap C = C$ or $\emptyset$. If $U \cap C = C$ then $C \subseteq U$ and so $Cl_E(C) \subseteq U$. but $Cl_E(C) = \overline{C} \cap E = E$. so $E \subseteq U$, so $U = E$. if $U \cap C = \emptyset$ then let $U' = E \setminus U$, so $U' \cap C = C$ and then $U' = E$ so $U = \emptyset$. $\square$

Corollary: Let X be a top sp let C be a component of X. C is closed

Thm Let X be a top. sp. Then TFAE:

(a) each component of X is open in X

(b) each point in X has a connected neighborhood in X.

<u>Corollary</u> Let X be open $\subseteq \mathbb{R}^d$. Components of X are open.

<u>Corollary</u> Let Y be a locally connected topological space. Let X be open in Y. then each component of X is open.

Something for you ...

<u>Theorem</u> Suppose $X = \cup\, \mathcal{U}$ for some disjoint set of open subsets of X. Then:

(a) each $U \in \mathcal{U}$ is clopen

(b) $\forall$ connected $C \subseteq X$, $\exists\, U \in \mathcal{U}$ s.t. $C \subseteq U$.

hence $\forall U \in \mathcal{U}$, $U = \cup \{[x] : x \in U\}$

(c) if each $U \in \mathcal{U}$ is connected then $\mathcal{U}$ is the set of connected components of X.

<u>Corollary</u> Suppose each point in X has a connected neighborhood. then there is a unique disjoint collection $\mathcal{U}$ of nonempty open connected subsets of X s.t. $X = \cup\, \mathcal{U}$.

<u>Corollary</u> Let X be open in $\mathbb{R}^d$. Then there is a unique disjoint collection $\mathcal{U}$ of non-empty open connected subsets of $\mathbb{R}^d$ s.t. $X = \cup\, \mathcal{U}$. Furthermore, $\mathcal{U}$ is countable.

<u>Proof that $\mathcal{U}$ is countable.</u> Let $(x_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}^d$.

Define $U \mapsto n_U$ on $\mathcal{U}$ by $n_U = \min\{\underbrace{n : x_n \in U}\}$ nonempty since U open in $\mathbb{R}^d$ & not empty.

so $\mathcal{U} \xrightarrow{n_U} \mathbb{N}$ is injective so $|\mathbb{N}| \geq |\mathcal{U}|$. $\square$

<u>Locally constant functions:</u>

<u>Theorem</u> Let X be a top. sp. let Y be a set, let $f : X \to Y$ be locally constant ($\forall p \in X$, $\exists\, U$ nbd of p in X, $\forall x \in U$, $f(x) = f(p)$). f is constant on each component of X.

Proof Let $Z = f[X]$. Let $\mathcal{U} = \{f^{-1}[\{y\}] : y \in Z\}$. Then $\mathcal{U}$ is a disjoint collection, each preimage is nonempty, and each is open since f is locally constant. hence each component of X is contained in some element of $\mathcal{U}$.

Corollary Let X be a top.sp. X is connected iff $\forall$ locally constant f, f is constant.

eg let X be a top sp. Let $f : X \longrightarrow \mathbb{C} \setminus \{0\}$ be cts, suppose g and h are branches of $\operatorname{Log} f$. ($g, h : X \to \mathbb{C}$, $e^g = f = e^h$).

Then $\forall x \in X$, $\exists\ k(x) \in \mathbb{Z}$ s.t. $h(x) = g(x) + 2\pi i k(x)$.

$k = \frac{g-h}{2\pi i}$ so k is cts. thus k is constant on each component of X. (locally constant).

Path Components

"path"

"pathwise connected" $\Longrightarrow$ connected

Define $\approx$ on X by $p_1 \approx p_2$ if $\exists$ path $\subseteq X$ starting at p_1 ending at p_2.

$[[p]] = \{x \in X : x \approx p\}$

Theorem Let X be a top.sp. in which each point has a pathwise connected nhd. then the path components of X are the same as the connected components of X. And each component of X is open.

Pf PC nhd of $x \Rightarrow$ PC_{omp}s are open (nhd $\subseteq PC_{omp}$).

Corollary let X be a top sp which is connected & locally pathwise connected then X is pathwise connected.

Corollary let X be a connected open subset of $\mathbb{R}^d$. X is path connected.

Compactness:

Compactness:

Defn Let X be a top.sp. We say X is compact if each open cover of X has a finite subcover. ($\forall$ collection $\mathcal{U}$ of open subsets of X, if $\cup\,\mathcal{U} = X$ then $\exists\, \mathcal{U}_0 \subseteq \mathcal{U}$ which is finite s.t. $\cup\,\mathcal{U}_0 = X$).

Borel (1894)

if $I_1, I_2, I_3, \ldots$ are intervals covering $[a,b]$ then $\sum_{k=1}^{\infty} \text{length}(I_k) \geq b-a$.