Lec 1/22

Monday, January 22, 2018 14:13

Frenet-Serret Apparatus

for each s, $T(s), N(s), B(s)$ forms an orthonormal, positively oriented basis for $\mathbb{R}^3$.

Since $|B(s)| \equiv 1$, $B'(s) \perp B(s)$.

$B'(s)$ lies in the plane spanned by $T(s)$ & $N(s)$.

$$B'(s) = \underbrace{\langle B'(s), T(s)\rangle}_{\text{we'll see that this is zero}} T(s) + \underbrace{\langle B'(s), N(s)\rangle}_{-\tau(s)} N(s).$$

The torsion $\tau(s)$ measures how fast $B(s)$ is spinning around $T(s)$.

eg let $r \in (0,\infty)$. define $\alpha: \mathbb{R} \to \mathbb{R}^3$ by $\alpha(s) = (r\cos(\frac{s}{r}), r\sin(\frac{s}{r}), 0)$.

then $\alpha'(s) = (-\sin(\frac{s}{r}), \cos(\frac{s}{r}), 0)$. $|\alpha'(s)| = 1$.

Thus α is unit speed and $T = \alpha'$.

$$T'(s) = -\frac{1}{r}(\cos(\tfrac{s}{r}), \sin(\tfrac{s}{r}), 0)$$

$$K(s) = |T'(s)| = \frac{1}{r}.$$

$$N(s) = \frac{T'(s)}{K(s)} = -(\cos(\tfrac{s}{r}), \sin(\tfrac{s}{r}), 0)$$

$$B(s) = T(s) \times N(s) = (0, 0, 1)$$

$$\tau(s) = -\langle B'(s), N(s)\rangle = -\langle 0, N(s)\rangle = 0.$$

Curves which move in a plane have 0 torsion.

eg unit speed circular helix

Let $r \in (0,\infty)$, $h \in \mathbb{R} \setminus \{0\}$. Let $\omega = \frac{1}{\sqrt{r^2+h^2}}$. Define $\alpha: \mathbb{R} \to \mathbb{R}^3$ by

$$\alpha(s) = (r\cos(\omega s), r\sin(\omega s), h\omega s)$$

$$\alpha'(s) = (-\omega r\sin(\omega s), r\omega\cos(\omega s), h\omega)$$

$$|\alpha'(s)| = \omega\sqrt{r^2+h^2} = 1.$$

Thus $T = \alpha'$

$$T'(s) = \left(-\omega^2 r \cos(\omega s),\ -\omega^2 r \sin(\omega s),\ 0\right)$$

$$\kappa(s) = |T'(s)| = \omega^2 r$$

$$\text{so } N(s) = \frac{T(s)}{\omega^2 r} = \left(-\cos(\omega s),\ -\sin(\omega s),\ 0\right)$$

Na
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$$B(s) = T(s) \times N(s) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\omega r \sin(\omega s) & \omega r \cos(\omega s) & h\omega \\ -\cos(\omega s) & -\sin(\omega s) & 0 \end{vmatrix}$$

$$= \left(h\omega \sin(\omega s),\ -h\omega \cos(\omega s),\ \omega r\right)$$

$$= \omega\left(h \sin(\omega s),\ -h\cos(\omega s),\ r\right)$$

$$B'(s) = \omega^2 h\left(\cos(\omega s), \sin(\omega s), 0\right) = -\omega^2 h\, N(s)$$

Thus $\tau(s) = \omega^2 h$

If $h > 0$, $\tau > 0$, and we say α is a "right-handed" circular helix.
if $h < 0$, $\tau < 0$, and we say α is a "left-handed" circular helix.

Prop Let $\underset{I \to \mathbb{R}^3}{\alpha}$ be a C^1 unit-speed curve in $\mathbb{R}^d$ with constant T.

then α is (part of) a straight line.

Pf Let $s_0 \in I$, $\overset{x_0 = \alpha(s_0)}{\text{Let}}$ $v = T(s_0)$. T const so $\forall s \in I$, $T(s) = v$.

but $T = \alpha'$ since α is unit speed so $\alpha(s) = \alpha(s_0) + \int_{s_0}^{s} \alpha'(\sigma)\, d\sigma$

$$= x_0 + v(s - s_0). \quad \blacksquare$$

Corollary Let α be a C^2 unit speed curve in $\mathbb{R}^d$ with curvature 0.
Then $T'(s) = 0$, so T is constant so α is part of a straight line

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The Frenet-Serret Theorem:

Let α be a C^3 unit-speed curve in $\mathbb{R}^3$ with Frenet-Serret apparatus (κ, τ, T, N, B). Suppose κ is never 0. Then for each s, we have

The Frenet-Serret equations

$$(*)\quad \begin{cases} T'(s) = & \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) & + \tau(s)B(s) \\ B'(s) = & -\tau(s)N(s) \end{cases}$$

$\Updownarrow$

$$(**)\quad \left\{ \begin{pmatrix} T' & N' & B' \end{pmatrix} = \begin{pmatrix} T & N & B \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{skew-symmetric}} \right.$$

(know this proof)

Proof That $T' = \kappa N$ follows by defn. of $N \overset{\text{def}}{=} \frac{T'}{\kappa}$.

Now $T' \perp T$ since $|T| = 1$ is constant. Hence $N \perp T$.

So $B \overset{\text{def}}{=} T \times N$ is orthogonal to T and N, and T, N, B is an ONB for $\mathbb{R}^3$.

Hence $N' = aT + bN + cB$ for suitable real-valued functions a, b, c.

Differentiating $0 = \langle T, N \rangle$ gives $0 = \langle T', N \rangle + \langle T, N' \rangle$

$= \langle \kappa N, N \rangle + \langle T, N' \rangle$

so $a = -\kappa$. $\qquad = \kappa + a$

Next $N' \perp N$ since $|N| = 1$. so $b = 0$. Differentiating $0 = \langle N, B \rangle$

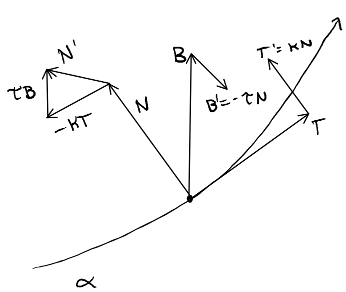
gives $0 = \langle N', B \rangle + \langle N, B' \rangle = c + -\tau$, so $c = \tau$.

Finally, differentiating $B = T \times N$ we get $B' = T' \times N + T \times N'$

so $B' = \kappa N \times N + T \times (-\kappa T + \tau B) = T \times \tau B = -\tau N$. $\square$

be proved by differentiating

Remark: $B = -\tau N$ can also be proved by differentiating inner products.



google frenet-serret

When does a curve lie in a plane?

Propn Let α be a C^3 unit-speed curve in $\mathbb{R}^3$ with κ never 0.

Then TFAE:

(a) α is a plane curve

(b) B is constant

(c) $\tau \equiv 0$

<u>Remark</u> The equivalence of (a) and (b) does not depend on F-S eqns.

<u>Proof</u> We have (b) $\Leftrightarrow$ (c) since $B' = -\tau N$.

(b) $\Rightarrow$ (a): Suppose B is constant. Then fix s_0 in the domain of α and let $x_0 = \alpha(s_0)$ and $n = B(s_0)$. Note $\forall s$, $B(s) = n$.

Then $0 = \langle \underbrace{\alpha(s_0) - x_0}_{0}, n \rangle$. And we have:

$$\frac{d}{ds} \langle \alpha(s) - x_0, n \rangle = \langle T(s), 0 \rangle = 0.$$

So $\langle \alpha(s) - x_0, n \rangle = 0 \ \forall s$. Thus $\alpha(s) - x_0$ is always $\perp$ to n and so α lies in the plane $\perp$ to n which intersects x_0.

$$\Pi = \{ x \in \mathbb{R}^3 : \langle x - x_0, n \rangle = 0 \}.$$

So α is a plane curve.

(a) $\Rightarrow$ (b): Suppose α is a plane curve.