~~eg~~ Let (S_n) be a simple symmetric RW on $\mathbb{Z}$.

Let $T = \inf\{n : S_n = 1\}$. We know $P(T<\infty) = 1$,

and $E(T) = \infty$. Let $Y = \inf\{S_n : n \le T\}$.

$Y > -\infty$ on $\{T<\infty\}$, so $P(Y>-\infty) = 1$.

We claim $E(Y) = -\infty$.

For each $\omega \in \{T<\infty\}$, $Y(\omega) = \min\{S_n(\omega) : n \le T(\omega)\}$

$= \min\{S_{T(\omega)\wedge n}(\omega) : n \ge 0\}$ (where $a \wedge b = \min\{a,b\}$).

$T\wedge n$ is a stopping time (it "doesn't involve looking into the future")

$E(T\wedge n) < \infty$. So $E(S_{T\wedge n}) = E(S_1)\cdot E(T\wedge \omega) = 0$ (for all n).

So $\forall \omega \in \{T<\infty\}$, $S_T(\omega) = 1$.

And $\lim_{n\to\infty} S_{T\wedge n}(\omega) = S_T(\omega)$.

So $S_T \longrightarrow 1$ almost surely.

For all n, $Y \le S_{T\wedge n} \le 1$.

So $|S_{T\wedge n}| \le 1 + |Y|$.

If $E(Y) > -\infty$, then $E(1+|Y|) < \infty$, So $E(S_{T\wedge n}) \to E(1) = 1$ by the dominated convergence theorem.

This contradicts the earlier finding that $E(S_{T\wedge n}) = 0 \ \forall n$,

So it must be the case that $E(Y) = -\infty$. $\square$

Wald's Second Eqn

Let (S_n) be a RW wrt a filtration $(\mathcal{F}_n)$.

Suppose $E(S_1^2) = \sigma^2 < \infty$ and $E(S_1) = 0$.

Let T be a stopping time with $E(T) < \infty$.

Then $E(S_T^2) = \sigma^2 E(T)$.

Lemma Let (Y_n) be an orthogonal sequence in L^2.

Suppose $\sum_{n=1}^{\infty} \underbrace{\|Y_n\|_2^2}_{= E(Y_n^2)} < \infty$, and $\sum_{n=1}^{\infty} Y_n$ converges a.s. to Y.

Then $Y \in L^2$, $\sum_{n=1}^{\infty} Y_n$ converges to Y in L_2,

and $\sum_{n=1}^{\infty} \|Y_n\|_2^2 = \|Y\|_2^2$.

Pf of Wald's Second Eqn

Let $X_n = S_n - S_{n-1}$ for all $n \geq 1$.

Let $X_n = S_n - S_{n-1}$ for all $n \geq 1$.

Let $Y_n = X_n 1_{\{T \geq n\}}$. Then $\sum_{n=1}^{\infty} Y_n$ converges to S_T pointwise on $\{T < \infty\}$ and hence a.s., since $E(T) < \infty$.

$$\|Y_n\|_2^2 = E(Y_n^2) = E\left[(X_n 1_{\{T \geq n\}})^2\right] = E(X_n^2 1_{\{T \geq n\}}).$$

Now $\{T \geq n\} = \Omega \setminus \{T \leq n-1\} \in \mathcal{F}_{n-1}$.

So X_n^2 and $1_{\{T \geq n\}}$ are independent.

So $\|Y_n\|_2^2 = P(T \geq n) \cdot E(S_1^2) = \sigma^2 P(T \geq n)$,

$$\text{So } \sum_{n=1}^{\infty} \|Y_n\|_2^2 = \sigma^2 \sum_{n=1}^{\infty} P(T \geq n)$$
$$= \sigma^2 E\left(\sum_{n=1}^{\infty} 1_{\{T \geq n\}}\right)$$
$$= \sigma^2 E(T).$$

In particular, $\sum_{n=1}^{\infty} \|Y_n\|_2^2 < \infty$.

Now if $m < n$, then
$$E(Y_m Y_n) = E(X_m 1_{\{T \geq m\}} X_n 1_{\{T \geq n\}})$$
$$= E(X_m X_n 1_{\{T \geq n\}}).$$

Now $X_m 1_{\{T \geq n\}}$ is $\mathcal{F}_n$-mble, so

X_n and $X_m 1_{\{T \geq n\}}$ are independent, so

$$E(X_m X_n 1_{\{T \geq n\}}) = E(X_m 1_{\{T \geq n\}}) \underbrace{E(X_n)}_{0} = 0.$$

Thus (Y_n) is an orthogonal sequence in L^2.

So, by the lemma, with $Y = S_T$, we have

$$E(S_T^2) = \|Y\|_2^2 = \sum_{n=1}^{\infty} \|Y_n\|_2^2 = \sigma^2 E(T). \qquad \square$$

(<u>Remark</u> More is true. It can be shown that

$$E\left(\sup_n |S_{T\wedge n}|^2\right) < \infty.$$)

<u>eg</u> Let (S_n) be a simple symmetric RW on $\mathbb{Z}$.

Let $a, b \in \mathbb{Z}$ with $a < 0 < b$. Let $N = \inf\{n : S_n \notin (a,b)\}$.

As we've seen, $E(N) < \infty$.

Of course, $E(S_1) = 0$ and $E(S_1^2) = 1$.

Hence $E(S_N^2) = E(N)$.

But $P(S_N = a) = \frac{b}{b-a}$ and $P(S_N = b) = \frac{a}{a-b}$,

So $E(N) = a^2 \frac{b}{b-a} + b^2 \frac{a}{a-b} = \frac{ab(a-b)}{b-a} = -ab$.

In particular, if $a = -b$, then $E(N) = b^2$.

It remains to prove the lemma.

First, let's prove the Riesz-Fischer Theorem.

The Riesz-Fischer Theorem (for $p=2$).

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let (f_n) be a Cauchy sequence in $L^2(\mu)$.

Then there exists $f \in L^2(\mu)$ such that $\|f - f_n\|_2 \longrightarrow 0$.

Pf By assumption, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $m, n \geq N$, $\|f_m - f_n\|_2 < \varepsilon$.

Hence there exist natural numbers $n_1 < n_2 < n_3 < \cdots$ such that for each k and for all $i, j \geq n_k$, $\|f_i - f_j\|_2 < 2^{-k}$.

Then for all $l > k$, $\|f_{n_k} - f_{n_l}\| < 2^{-k}$.

It suffices to show that there exists $f \in L^2$ such that $\|f - f_{n_k}\|_2 \longrightarrow 0$ as $k \longrightarrow \infty$.

Let $g_k = f_{n_k}$. Let $h = |g_1| + \sum_{k=2}^{\infty} |g_k - g_{k-1}|$

and $h_n = |g_1| + \sum_{k=2}^{n} |g_k - g_{k-1}|$.

$$\begin{aligned}
\|h_n\|_2 &\leq \|g_1\|_2 + \sum_{k=2}^{\infty} \|g_k - g_{k-1}\|_2 \\
&\leq \|g_1\|_2 + \sum_{k=2}^{\infty} 2^{-(k-1)} \\
&= \underbrace{\|g_1\|_2 + 1}_{\text{does not depend on } n} < \infty.
\end{aligned}$$

$h_n \uparrow h$, so $h_n^2 \uparrow h^2$, so $\int h_n^2 \, d\mu \uparrow \int h^2 d\mu$, so $\|h_n\|_2 \uparrow \|h\|_2$.

So $\|h\|_2 \le \|g_1\|_2 + 1 < \infty$.

Hence $\int h^2 \, d\mu = \|h\|_2^2 < \infty$. Thus $h < \infty$ almost everywhere.

In other words $|g_1| + \sum_{k=2}^{\infty} |g_k - g_{k-1}| < \infty$ a. e.

Let $W = \{h < \infty\}$. $\mu(X \setminus W) = 0$ and for each $x \in W$,

$|g_1(x)| + \sum_{k=0}^{\infty} |g_k(x) - g_{k-1}(x)| < \infty$, so $g_1(x) + \sum_{k=0}^{\infty} (g_k(x) - g_{k-1}(x))$ converges,

But $g_1(x) + \sum_{k=0}^{n} (g_k(x) - g_{k-1}(x)) = g_n(x)$.

Thus for each $x \in W$, $\lim_{n\to\infty} g_n(x)$ exists and is finite.

Define f on X by

$$f(x) = \begin{cases} \lim_{n\to\infty} g_n(x) & x \in W \\ 0 & x \in X \setminus W. \end{cases}$$

Then $g_n \longrightarrow f$ a.e.

Now $|g_n| \le |g_1| + \sum_{k=2}^{n} |g_k - g_{k-1}| \le h$,

and $|f| \le h$, so $|f - g_n|^2 \le (|f| + |g_n|)^2 \le (h+h)^2 = 4h^2$.

Now $\int 4h^2 \, d\mu < \infty$ and $|f - g_n|^2 \longrightarrow 0$ a.e.,

and so $\int |f - g_n|^2 \, d\mu \longrightarrow \int 0 \, d\mu = 0$ by the dominated convergence Theorem. In other words,

$\|f - g_n\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. $\square$

Corollary Let (f_n) be an orthogonal sequence in $L^2(\mu)$. Suppose $\sum_{n=1}^{\infty} \|f_n\|_2^2 < \infty$, and that $\sum_{n=1}^{\infty} f_n$ converges a.e. to f.

Then $f \in L^2(\mu)$, $\sum_{n=1}^{\infty} f_n \longrightarrow f$ in $L^2(\mu)$,

and $\sum_{n=1}^{\infty} \|f_n\|_2^2 = \|f\|_2^2$.