4/5

Friday, April 5, 2019 14:30

<u>Propn</u> Let $(\mathcal{F}_n)$ be a filtration.

Let $A_n \in \mathcal{F}_n$ for each n.

Define $N : \Omega \longrightarrow [0,\infty]$ by

$N(\omega) = \inf\{n : \omega \in A_n\}$

Then N is a stopping time.

pf $N(\omega) \leq n$ iff for some $m \leq n$, $\omega \in A_m$.

Thus $\{N \leq n\} = \bigcup_{m=0}^{n} A_m$.

For each $m \leq n$, $A_m \in \mathcal{F}_m \subseteq \mathcal{F}_n$,

So $A_m \in \mathcal{F}_n$. Thus $\{N \leq n\} \in \mathcal{F}_n$. $\square$

eg Let (S_n) be a sequence of RVs in a mble space $(E, \mathcal{E})$. Suppose (S_n) is <u>adapted</u> to a filtration $(\mathcal{F}_n)$

(This means that for each n, S_n is $\mathcal{F}_n/\mathcal{E}$-mble)

Let $A \in \mathcal{E}$. Let $N = \inf\{n : S_n \in A\}$.

Then N is a stopping time

[Pf: $N(\omega) = \inf\{n : \omega \in A_n\}$ where $A_n = \{S_n \in A\} \in \mathcal{F}_n$]

Let $B \in \mathcal{E}$. Let $T = \inf\{n \geq N : S_n \in B\}$.

Then T is a stopping time.

[Pf: $T(\omega) = \inf\{n : N(\omega) \geq n \text{ and } S_n(\omega) \in B\}$

$= \inf\{n : \omega \in B_n\}$,

where $B_n = \underbrace{\{N \leq n\}}_{\mathcal{F}_n} \cap \underbrace{\{S_n \in B\}}_{\mathcal{F}_n} \in \mathcal{F}$].

et cetera...

eg Let (S_n) be a random walk in $\mathbb{R}$ wrt a filtration $(\mathcal{F}_n)$.

Let $-\infty < a < 0 < b < \infty$. Let $N = \inf\{n : S_n \notin (a,b)\}$.

Then N is a stopping time.

[Pf: $N = \inf\{n : S_n \in A\}$ where $A = \mathbb{R} \setminus (a,b)$.]

Wald's First Equation

Let (S_n) be a RW in $\mathbb{R}$ wrt a filtration $(\mathcal{F}_n)$.

Assume $E|S_1| < \infty$ (i.e. assume (S_n) has integrable increments).

Let N be a stopping time with $E(N) < \infty$.

Then $E(S_N) = E(S_1) \cdot E(N)$ $\quad \left(E(X_1 + \cdots + X_N) = E(N) \cdot E(X_1) \right)$.

Pf of Course $X_n = S_n - S_{n-1}$ for $n \geq 1$, and $S_0 = 0$.

$X_1, X_2, X_3, \ldots$ are iid, adapted to $(\mathcal{F}_n)$,

and for each n, $\mathcal{F}_n$ and $\sigma(X_{n+1}, X_{n+2}, \ldots)$ are independent.

<u>Case 1</u> Suppose each $X_n \geq 0$. (Then we don't care whether X_n is integrable, and we don't care whether $E(N) < \infty$, or even whether $P(N < \infty) = 1$.)

Then $S_N = X_1 + \cdots + X_N$.

$$= \sum_{n \geq 1} X_n 1_{\{n \leq N\}}$$

So
$$E(S_n) = \sum_{n \geq 1} E(X_n 1_{\{n \leq N\}})$$

$$= \sum_{n \geq 1} E(X_n 1_{\underbrace{\{N \leq n-1\}^c}_{\in \mathcal{F}_{n-1}}})$$

$\mathcal{F}_{n-1}$, so indep of X_n

$$= \sum_{n \geq 1} E(X_n)\, P(n \leq N)$$

—

$$= E(S_1) \sum_{n \geq 1} P(n \leq N)$$

$$= E(S_1) \sum_{n \geq 1} n P(N = n)$$

$$= E(S_1) \cdot E(N) .$$

<u>Case 2</u> The General Case.

$$S_N = X_1 + \cdots + X_N$$

$$= (X_1^+ + \cdots + X_N^+) - (X_1^- + \cdots + X_N^-) .$$

For each n, $X_1^+, \ldots, X_n^+$ are $\mathcal{F}_n$-measurable,

and $X_{n+1}^+, X_{n+2}^+, \ldots$ are $\sigma(X_{n+1}, X_{n+2}, \ldots)$-measurable,

So $\mathcal{F}_n$ and $\sigma(X_{n+1}^+, X_{n+2}^+, \ldots)$ are independent.

So $E(X_1^+ + \cdots + X_N^+) = E(X_1^+) E(N)$ by case 1.

Similarly, $E(X_1^- + \cdots + X_N^-) = E(X_1^-) E(N)$ also by case 1.

So, assuming that $E|X_1| < \infty$ and $E|N| < \infty$, we have $E(X_1^+) < \infty$ and $E(X_1^-) < \infty$ and $P(N < \infty) = 1$. So

$$E(S_N) = E(X_1^+) E(N) - E(X_1^-) E(N)$$

$$= E(S_1) \cdot E(N) .$$

$\square$

eg Let (S_n) be a RW wrt a filtration $(\mathcal{F}_n)$, and Suppose $E|S_1| < \infty$ and $E(S_1) = 0$.

Let N be a stopping time with $E(N) < \infty$.

Then $E(S_N) = 0$.

[Pf $E(S_N) = E(S_1) \cdot E(N) = 0 \cdot E(N) = 0$].

eg Let (S_n) be a symmetric simple RW on $\mathbb{Z}$. (steps are ± 1.)

Let $a, b \in \mathbb{Z}$ with $a < 0 < b$. Let $N = \inf\{n : n \notin (a,b)\}$.

Since (S_n) is nondegenerate, $E(N) < \infty$, so $P(N < \infty) = 1$. For each ω, if $N(\omega) < \infty$, then

$$S_N(\omega) \in \{a, b\}, \quad \text{because } S_{N-1}(\omega) \in (a,b), \text{ and } S_N(\omega) \notin (a,b),$$
$$\text{and } |S_N(\omega) - S_{N-1}(\omega)| = 1.$$

$S_N(\omega) = S_{N(\omega)}(\omega)$

By Wald's first equation, $E(S_N) = 0$ because $E(S_1) = 0$. But since $P(S_n \text{ is a or b}) = 1$,

$E(S_n) = aP(S_n = a) + bP(S_n = b)$.

Let $\alpha = P(S_N = a)$, $\beta = P(S_N = b)$.

Then $\alpha + \beta = 1$ and $a\alpha + b\beta = 0$.

So $\beta = 1 - \alpha$, $a\alpha + b(1-\alpha) = 0$.

So $(a-b)\alpha + b = 0$, so $\alpha = \frac{b}{b-a}$.

so $\beta = \frac{a}{a-b}$.

Let $T = \inf\{n : S_n = b\}$.

Then $P(T < \infty) \geq P(N < \infty \text{ and } S_N = b)$

$$= P(S_N = b) = \beta = \frac{a}{a-b} \longrightarrow 1 \text{ as } a \to -\infty.$$

Since a was arbitrary here, this means $P(T < \infty) = 1$.

For each ω, if $T(\omega) < \infty$, then $S_T(\omega) = b$.

Hence $E(T) = \infty$ because if $E(T) < \infty$, then

Wald's First Equation says

$$E(S_T) = E(S_1)\,E(T) = 0 \neq b,$$

which is contradictory.