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Random Walk

Let $X_1, X_2, X_3, \dots$ be an iid sequence of real RVs.

Let $S_0 = 0$, $S_1 = X_1$, $S_2 = X_1 + X_2$, $\dots$

The sequence (S_n) is called a random walk in $\mathbb{R}$.

Eg Suppose $S_n = \sum_{k \le n} X_k$ where $X_1, X_2, \dots$ are indep. and $P(X_k = 1) = \frac{1}{2} = P(X_k = -1)$. Then (S_n) is called a symmetric simple RW on $\mathbb{Z}$.

eg In the previous example, if instead $P(X_k = 1) = p$ and $P(X_k = -1) = 1 - p$ where $0 < p < 1$, $p \neq \frac{1}{2}$.

Then (S_n) is called an asymmetric simple RW on $\mathbb{Z}$.

Defn A RW (S_n) is called non-degenerate when

$$P(X_1 \neq 0) > 0 \qquad (X_n = S_n - S_{n-1})$$

Theorem Let (S_n) be a non-degenerate RW on $\mathbb{R}$.

Let $-\infty < a < 0 < b < \infty$. Let $N = \inf\{n : S_n \notin (a,b)\}$

$(N(\omega) = \inf\{n : S_n(\omega) \notin (a,b)\})$.

Remember $\inf \emptyset = \infty$.

Then N is mble and $E(N) < \infty$.

in particular, $P(N < \infty) = 1$.

pf $\{N > n\} = \{S_1 \in (a,b), \dots, S_n \in (a,b)\}$.

$\in \sigma(S_1, \dots, S_n)$

$= \sigma(S_k^{-1}[A] : A \in \text{Borel}(\mathbb{R}),\ k \in \{1, \dots, n\})$.

$\subseteq \mathcal{F}$.

This holds for $n = 1, 2, 3, \dots$, and $\{N > 0\} = \Omega$ since $S_0 \equiv 0$

$N : \Omega \longrightarrow \{1, 2, 3, \dots, \infty\}$. Thus N is mble and

in fact, $\{N > n\}$ depends only on $S_1, \dots, S_n$.

Now let's show $E(N) < \infty$.

Either $P(X_1 > 0) > 0$ or $P(X_1 < 0) > 0$.

The two cases are similar, so let's just

Consider the case where $P(X_1 > 0) > 0$.

$\{X_1 \geq \frac{1}{\ell}\} \uparrow \{X_1 > 0\}$, so $P(X_1 \geq \frac{1}{m}) \uparrow P(X_1 > 0)$.

$(\ell \in \mathbb{N})$

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$(\ell \in \mathbb{N})$

So $\exists\, \varepsilon > 0$ s.t. $P(X_1 \geq \varepsilon) > 0$.

Choose $m \in \mathbb{N}$ such that $m\varepsilon \geq b - a$.

Then for each $x \in (a,b)$,

$$
\begin{aligned}
P(x + S_m \notin (a,b)) &\geq P(S_m \geq b-a) \\
&\geq P(X_1 \geq \varepsilon, \ldots, X_m \geq \varepsilon) \\
&= P(X_1 \geq \varepsilon) \cdot \cdots \cdot P(X_m \geq \varepsilon) \\
&= [P(X_1 \geq \varepsilon)]^m
\end{aligned}
$$

Hence
$$
\begin{aligned}
P(N > m) &= P(S_k \in (a,b) \text{ for } k = 1, \ldots, m) \\
&\leq P(S_m \in (a,b)) \\
&= 1 - P(S_m \notin (a,b)) \\
&\leq 1 - [P(X_1 \geq \varepsilon)]^m
\end{aligned}
$$

Now for $n = 1, 2, 3, \ldots$

$$
\begin{aligned}
P(N > (n+1)m) &= P(N > (n+1)m,\ N > nm) \\
&\leq P(S_{(n+1)m} \in (a,b),\ N > nm) \\
&= P(N > nm) - P(S_{(n+1)m} \notin (a,b),\ \underbrace{N > nm}_{\Downarrow\ S_{nm} \in (a,b)})
\end{aligned}
$$

mble wrt $\sigma(S_1, \ldots, S_{nm}) = \sigma(X_1, \ldots, X_{nm})$

$$\leq P(N>nm) - P(X_{nm+1} \geq \varepsilon, \ldots, X_{nm+m} \geq \varepsilon, N>nm)$$

depends only on $X_1, \ldots, X_{nm}$

$$= P(N>nm) - P(X_{nm+1} \geq \varepsilon, \ldots, X_{(n+1)m} \geq \varepsilon) \cdot P(1 \quad > nm)$$

$$= P(N>nm)\left(1 - P(X_1 \geq \varepsilon)^m\right)$$

Hence by induction, $P(N>nm) \leq \left(1 - P(X_1 \geq \varepsilon)^m\right)^n$

Hence $\frac{1}{m} E(N) = E\left(\frac{N}{m}\right) \leq \sum_{n=0}^{\infty} P\left(\frac{N}{m} > n\right)$ ← exercise 26.

$$= \sum_{n=0}^{\infty} P(N>nm)$$

$$\leq \sum_{n=0}^{\infty} \left(1 - P(X_1 \geq \varepsilon)^m\right)^n \quad \leftarrow \text{geometric sum}$$

$$= \frac{1}{P(X_1 \geq \varepsilon)^m}$$

$$< \infty \qquad \square$$

Filtrations

A *filtration* is an increasing sequence $(\mathcal{F}_n)_{n \geq 0}$ of sub-σ-fields of $\mathcal{F}$.

eg Let $X_1, X_2, X_3, \ldots$ be RVs. Let $\mathcal{F}_n = \sigma(X_k : k \leq n)$ for $n = 0, 1, 2, \ldots$

(so $\mathcal{F}_0 = \{\emptyset, \Omega\}$).

Then $(\mathcal{F}_n)$ is a filtration.

If each X_k is Real-Valued and $S_n = \sum_{k \le n} X_k$ for $n=0,1,2,\dots$,

then $\mathcal{F}_n$ is also equal to $\sigma(S_0, S_1, \dots, S_n)$.

$(\mathcal{F}_n)$ is called the natural filtration of $(X_n)_{n\geq 1}$ or of $(S_n)_{n\geq 0}$.

Random walks with respect to a filtration

Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. Let $X_n : \Omega \to \mathbb{R}$

be $\mathcal{F}_n$-mble for each $n \geq 1$.

Suppose also that for each $n\geq 0$, $\mathcal{F}_n$ and $\sigma(X_{n+1}, X_{n+2}, \dots)$

are independent.

Assume also that $X_1, X_2, X_3, \dots$ are identically distributed

and let $S_n = \sum_{k \le n} X_k$ for $n=0,1,2,\dots$ Then we

say (S_n) is a RW wrt $(\mathcal{F}_n)$.

eg Let S_n be a RW in the previous sense.

Let $(\mathcal{F}_n)$ be its natural filtration.

Then (S_n) is a RW wrt $(\mathcal{F}_n)$.

eg Let (S_n) be a RW in the previous sense.

Let Y be a RV independent of (S_n).

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n, Y)$ for each $n \geq 0$.

Then (S_n) is a RW wrt $(\mathcal{F}_n)$.

<u>Stopping Times</u> Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration.

To say that N is a <u>stopping time</u> wrt $(\mathcal{F}_n)$ means that $N : \Omega \to \{0, 1, 2, \dots, \infty\}$, and

$\{N \leq n\} \in \mathcal{F}_n$ for $n = 0, 1, 2, \dots$.

could also use these

$\{N > n\}$

$\{N = n\} = \{N \leq n\} \setminus \{N \leq n-1\}$

$\{N \leq n\} = \dot{\bigcup}_{k=0}^{n} \{N = k\}$