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(S_n) = Symmetric Simple RW on $\mathbb{Z}$.

Thm Let $n \in \{1, 2, 3, \dots\}$. Then $P(S_1 \neq 0, \dots, S_{2n} \neq 0) = P(S_{2n} = 0)$.

Pf Done last time.

Corollary Let R be the time of the first return to 0:

$$R = \inf\{m \geq 1 : S_m = 0\}.$$

Then $P(R > 2n) \sim \frac{1}{\sqrt{\pi n}}$ as $n \longrightarrow \infty$.

Pf $P(R > 2n) = P(S_1 \neq 0, \dots, S_{2n} \neq 0) = P(S_{2n} = 0)$

$$= \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n}$$

$$= 2^{-2n} \binom{2n}{n}$$

$$= 2^{-2n} \frac{(2n)!}{(n!)^2}$$

$$= 2^{-2n} \frac{\sqrt{2\pi}\,(2n)^{2n+\frac{1}{2}} e^{-2n} e^{\frac{\theta_{2n}}{12\cdot 2n}}}{\left(\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{\theta_n}{12n}}\right)^2} \qquad \text{where } \theta_n, \theta_{2n} \in (0,1).$$

$$= \frac{2^{-2n} \sqrt{2\pi}\, 2^{2n} n^{2n} \sqrt{2}\sqrt{n}\, e^{-2n} e^{\frac{\theta_{2n}}{24n}}}{2\pi\, n^{2n}\, n\, e^{-2n} e^{\frac{2\theta_n}{12n}}}$$

$$= \frac{1}{\sqrt{\pi n}} \underbrace{e^{\left(\frac{\theta_{2n}}{24n} - \frac{\theta_n}{6n}\right)}}_{\substack{\longrightarrow 1 \\ \text{as } n \longrightarrow \infty}} \qquad \square$$

<u>Notation</u> $L_{2n} = \max \underbrace{\{m \le 2n : S_m = 0\}}_{\neq \emptyset \text{ because } S_0 = 0.}$.

(L_{2n} is <u>not</u> a stopping time, because it depends on the future).

<u>Propn</u> Let $k \in \{0, 1, 2, \ldots, n\}$. Then

$$P(L_{2n} = 2k) = P(S_{2k} = 0)\, P(S_{2n-2k} = 0).$$

<u>Pf</u>
$$\begin{aligned} P(L_{2n} = 2k) &= P(S_{2k} = 0, S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0) \\ &= P(S_{2k} = 0, S_{2k+1} - S_{2k} \neq 0, \ldots, S_{2n} - S_{2k} \neq 0) \\ &= P(S_{2k} = 0)\, P(S_{2k+1} - S_{2k} \neq 0, \ldots, S_{2n} - S_{2k} \neq 0) \\ &= P(S_{2k} = 0)\, P(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0) \\ &= P(S_{2k} = 0)\, P(S_{2n-2k} = 0). \qquad \square \end{aligned}$$

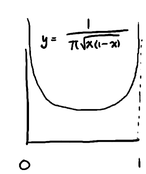
Remark It follows that the distribution of L_{2n} in $\{0, 2, \ldots, 2n\}$ is symmetric about n.

Thus if two people were to bet \$1 on a coin toss every day for a year, then, with probability at least $\frac{1}{2}$, one of them would be ahead from early July to the end of the year, an event that would surely cause the other player to complain about his bad luck. (In a non-leap year, the months January through June have 181 days, while July through December have 184 days).

The Arcsine Law for the last visit to 0

Let $0 < a < b < 1$. Then as $n \to \infty$,

$$P\left(a \le \frac{L_{2n}}{2n} \le b\right) \longrightarrow \int_a^b \frac{1}{\pi\sqrt{x(1-x)}}\,dx$$



$$= \frac{2}{\pi}\left(\arcsin\sqrt{b} - \arcsin\sqrt{a}\right).$$

Pf Let $K_n = \{k \in \{1, \ldots, n\} : a < \frac{2k}{2n} < b\}$.

Then as $n \longrightarrow \infty$, uniformly for $k \in K_n$,

we have $$P(L_{2n} = 2k) = P(S_{2k} = 0) P(S_{2n-2k} = 0)$$

$$\sim \frac{1}{\sqrt{\pi k}} \cdot \frac{1}{\sqrt{\pi (n-k)}}$$

$$= \frac{\frac{1}{n}}{\pi \sqrt{\frac{k}{n}\left(1 - \frac{k}{n}\right)}}$$

$$= \frac{1}{\pi \sqrt{x_{nk}(1 - x_{nk})}} (x_{nk} - x_{n,k-1})$$

where $x_{nk} = \frac{k}{n}$. Therefore,

$$P\left(a \leq \frac{L_{2n}}{2n} \leq b\right) = \sum_{k \in K_n} P(L_{2n} = 2k)$$

$$\sim \sum_{k \in K_n} \frac{1}{\pi \sqrt{x_{nk}(1 - x_{nk})}} (x_{nk} - x_{n,k-1})$$

$$\longrightarrow \int_a^b \frac{1}{\pi \sqrt{x(1-x)}} \, dx \qquad (*).$$

Substitute $y = \sqrt{x}$ so $y^2 = x$, $2y\,dy = dx$

$$a \xrightarrow{x} b$$
$$\sqrt{a} \xrightarrow{y} \sqrt{b}.$$

$$(*) = \int^{\sqrt{b}} \frac{1}{\pi} \frac{1}{y} \frac{1}{\sqrt{1-y^2}} \, 2y\,dy$$

$\sqrt{a}$

$$= \frac{2}{\pi} \int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\sqrt{1-y^2}} \, dy$$

$$= \frac{2}{\pi} \left(\arcsin\sqrt{b} - \arcsin\sqrt{a} \right). \qquad \square$$

Time Above 0

Let $(0, s_0), \ldots, (2n, s_{2n})$ be a path of even length $2n$.

Let $Z = \{m : s_m = 0\}$. Let $m_0, \ldots, m_z$ be an enumeration of Z in increasing order.

$m_0 = 0$ and for $j = 1, \ldots, z$, $m_j - m_{j-1}$ is even and ≥ 2.

For each $j \in \{1, \ldots, z\}$, either $s_m > 0$ for each $m \in \{m_{j-1}+1, \ldots, m_j - 1\}$, or $s_m < 0$ for each $m \in \{m_{j-1}+1, \ldots, m_j - 1\}$.

The time that the path is at or above 0 is

$$\left| \{ m \in \{1, \ldots, 2n\} : s_{m-1} \geq 0 \text{ and } s_m \geq 0 \} \right|.$$

Let $\Pi_{2n} = \left| \{ m \in \{1, \ldots, 2n\} : S_{2m-1} \geq 0 \text{ and } S_{2m} \geq 0 \} \right|.$

$=$ The time that $(S_m)_{m=0}^{2n}$ is at or above 0.

Theorem For $k = 0, 1, \ldots, n$, $P(\Pi_{2n} = 2k) = P(L_{2n} = 2k)$.

Changes of Sign

Theorem The probability that up to time $2n+1$, (S_m) undergoes exactly r changes of sign is $2P(S_{2n+1} = 2r+1)$.

Note: this probability decreases as r increases.

r	$2P(S_{99} = 2r+1)$
0	0.1592
1	0.1529
2	0.1412
$\vdots$	$\vdots$
13	0.0040
$\vdots$	$\vdots$
49	2^{-98} (check)