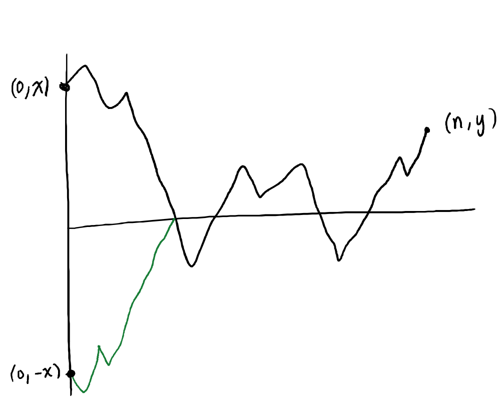
The Reflection Principle (Desiré André 1887)

Let x, y be integers > 0. Then the number of paths from $(0, x)$ to (n, y) that are 0 at some time is equal to the total number of paths from $(0, -x)$ to (n, y).

Pf



Let W be the set of all paths $w = ((0, w_0), (1, w_1), \ldots, (n, w_n))$ such that $w_k = 0$ for some $k \in \{1, \ldots, n\}$. for each such $w \in W$, let $K(w) = \min\{k : w_k = 0\}$, and let

$R(w) = ((0, w_0'), (1, w_1'), \ldots, (n, w_n'))$, where

$$w_k' = \begin{cases} -w_k & \text{if } k < K(w) \\ w_k & \text{if } k \geq K(w) \end{cases}$$

Then $R : W \longrightarrow W$, because for each such w, $w_k' - w_{k-1}' = \begin{cases} -(w_k - w_{k-1}) & \text{if } k \leq K(w) \\ \ldots \end{cases}$

$w_k \quad w_{k-1} \qquad \text{if } k > K(w)$

which is ± 1 in all cases.

Also, $K(R(w)) = K(w)$. So $R(R(w)) = w$.

So R is a bijection $W \longleftrightarrow W$.

Let $U = \{w \in W : w_0 = x,\ w_n = y\}$

and $V = \{w \in W : w_0 = -x,\ w_n = y\}$

Then $R[U] \subseteq V$ and $R[V] \subseteq U$, so in fact, since $R^2 = id_W$.

$R|_U$ is a bijection $U \longleftrightarrow V$.

In particular, $|U| = |V|$. $\square$

The Ballot Theorem Consider an election with two candidates A and B. Suppose candidate A gets α votes and B gets β votes, where $\alpha > \beta$. Then the probability that throughout the counting process, A leads is $\frac{\alpha - \beta}{\alpha + \beta}$.

Pf Think of the ballots as numbered from 1 to $\alpha + \beta$, where those numbered from 1 to α are for A and those numbered from $\alpha + 1$ to $\alpha + \beta$ are for B. The set of possible outcomes of the counting process may be thought of as the set of permutations of $\{1, \ldots, \alpha + \beta\}$, of which there are $(\alpha+\beta)!$. To each such outcome, there corresponds a path from $(0,0)$ to $(\alpha+\beta, \alpha-\beta)$, and to each such path there correspond $\alpha!\beta!$ outcomes.

Thus all such paths are equally likely.

The number of such paths for which A leads throughout is equal to the number of paths from $(1,1)$ to $(\alpha+\beta, \alpha-\beta)$ which are never 0.

By the reflection principle, the number of paths from $(1,1)$ to $(\alpha+\beta, \alpha-\beta)$ which are 0 at some time is equal to the number of paths from $(1,-1)$ to $(\alpha+\beta, \alpha-\beta)$.

Hence the number of paths from $(1,1)$ to $(\alpha+\beta, \alpha-\beta)$ which are never 0 is

$$\underbrace{\binom{\alpha+\beta-1}{\alpha-1}}_{\substack{\alpha+\beta-1 \text{ steps} \\ \alpha-1 \text{ steps of } +1 \\ \beta \text{ steps of } -1}} - \underbrace{\binom{\alpha+\beta-1}{\alpha}}_{\substack{\alpha+\beta-1 \text{ steps} \\ \alpha \text{ steps of } +1 \\ \beta-1 \text{ steps of } -1}} = \binom{\alpha+\beta-1}{\alpha-1} - \binom{\alpha+\beta-1}{\alpha-1}\frac{(\alpha+\beta-1)-\alpha+1}{\alpha}$$

(Recall $\underbrace{\binom{n}{k} = \binom{n}{k-1}\frac{n-k+1}{k}}_{\text{useful for computing } \binom{n}{k}\,!}$)

not factorial ↓

$$= \binom{\alpha+\beta-1}{\alpha-1}\left(1 - \frac{\beta}{\alpha}\right)$$

$$= \binom{\alpha+\beta-1}{\alpha-1}\left(\frac{\alpha-\beta}{\alpha}\right)$$

The total number of paths from $(0,0)$ to $(\alpha+\beta, \alpha-\beta)$ is

$$\binom{\alpha+\beta}{\alpha} = \binom{\alpha+\beta-1}{\alpha-1} + \binom{\alpha+\beta-1}{\alpha} = \binom{\alpha+\beta-1}{\alpha-1} + \binom{\alpha+\beta-1}{\alpha-1}\frac{(\alpha+\beta-1)-\alpha+1}{\alpha}$$

$$= \Big(\alpha+\beta-1 \qquad \Big) \Big(\alpha+\beta-1$$

$\alpha - 1$ / α

$$= \binom{\alpha+\beta-1}{\alpha-1}\left(1+\frac{\beta}{\alpha}\right) = \binom{\alpha+\beta-1}{\alpha-1}\left(\frac{\alpha+\beta}{\alpha}\right)$$

$$= \frac{(\alpha+\beta)(\alpha+\beta-1)!}{\alpha\,(\alpha-1)!\,\beta!} = \binom{\alpha+\beta-1}{\alpha-1}\left(\frac{\alpha+\beta}{\alpha}\right).$$

So the probability that A leads throughout

is $$\frac{\binom{\alpha+\beta-1}{\alpha-1}\left(\frac{\alpha-\beta}{\alpha}\right)}{\binom{\alpha+\beta-1}{\alpha-1}\left(\frac{\alpha+\beta}{\alpha}\right)} = \frac{\alpha-\beta}{\alpha+\beta}. \qquad \square$$

Let (S_n) be a symmetric simple RW on $\mathbb{Z}$.

<u>Theorem</u>: Let $n \in \{1,2,3,\dots\}$. Then $P(S_1 \neq 0, \dots, S_{2n} \neq 0) = P(S_{2n} = 0)$.

Pf By symmetry, $P(S_1 > 0, \dots, S_{2n} > 0) = P(S_1 < 0, \dots, S_{2n} < 0)$.

($(-S_n)$ is also a symmetric simple RW on $\mathbb{Z}$).

As we know, if $j < l$ and $S_j(\omega) > 0$, $S_l(\omega) < 0$ (or vice versa) then $S_k(\omega) = 0$ for some k between j & l.

Hence the union of the disjoint events $\{S_1 > 0, \dots, S_{2n} > 0\}$ and $\{S_1 < 0, \dots, S_{2n} < 0\}$ is the event $\{S_1 \neq 0, \dots, S_{2n} \neq 0\}$. Hence

$$P(S_1 \neq 0, \dots, S_{2n} \neq 0) = 2P(S_1 > 0, \dots, S_{2n} > 0)$$

$$= \sum_{r=1}^{n} P(S_1 > 0, \ldots, S_{2n-1} > 0,\ S_{2n} = 2r).$$

The number of paths from $(0,0)$ to $(2n, 2r)$ that are never 0 at times ≥ 1 is equal to the number of paths from $(1,1)$ to $(2n, 2r)$ that are never 0, and this is equal to $K_r - J_r$ where K_r is the number of paths from $(1,1)$ to $(2n, 2r)$ and J_r is the number of such paths that are zero at some time. By the reflection principle, J_r is the number of paths from $(1,-1)$ to $(2n, 2r)$. So J_r is the number of paths from $(1,1)$ to $(2n, 2(r+1))$. In other words, $J_r = K_{r+1}$. So the number of paths from $(0,0)$ to $(2n, 2r)$ that are never 0 at times ≥ 1 is $K_r - K_{r+1}$.

Hence $P(S_1 > 0, \ldots, S_{2n-1} > 0,\ S_{2n} = 2r) = \dfrac{K_r - K_{r+1}}{2^{2n}}$.

Thus
$$P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = 2 \sum_{r=1}^{n} P(S_1 > 0, \ldots, S_{2n-1} > 0,\ S_{2n} = 2r)$$
$$= 2 \sum_{r=1}^{n} \frac{K_r - K_{r+1}}{2^{2n}}$$
$$= \frac{2}{2^{2n}} \left(K_1 - \underbrace{K_{n+1}}_{=0}\right).$$
$$= \frac{2K_1}{2^{2n}}$$
$$= \frac{1}{2^{2n-1}} \binom{2n-1}{\cdot}$$

$$2 \quad n$$

2n−1 steps
n steps of +1
n−1 steps of −1

$$= P(S_{2n-1} = 1) .$$

$$P(S_{2n} = 0) = P(S_{2n-1} = 1, S_{2n} = 0) + P(S_{2n-1} = -1, S_{2n} = 0)$$

$$= \frac{1}{2} P(S_{2n-1} = 1) + \frac{1}{2} P(S_{2n-1} = -1)$$

$$= P(S_{2n-1} = 1)$$

$$= P(S_1 \neq 0, \ldots, S_{2n} \neq 0) . \qquad \square$$