(S_n) = asymmetric simple RW on $\mathbb{Z}$

$P(S_1 = 1) = p$

$P(S_1 = -1) = q = 1-p$

$p > \frac{1}{2} > q$

$\varphi(x) = \left(\frac{q}{p}\right)^x.$

$\varphi(S_n)$ is a mtgle.

$T_x = \inf\{n : S_n = x\}$

$(x \in \mathbb{Z})$

$S_n \longrightarrow \infty$ a.s. by strong law of large #s for Bernoulli RVs.

If $a, b \in \mathbb{Z}$ with $a < 0 < b$

Then $P(T_a < T_b) = \dfrac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$

If $b \in \mathbb{Z}$ and $b > 0$, then $T_b < \infty$ a.s.

If $a \in \mathbb{Z}$ and $a < 0$ then $P(T_a < \infty) = \left(\frac{q}{p}\right)^{-a}.$

<u>Remark</u>

Remark

$$\{T_a < \infty\} = \{\inf_n S_n \leq a\}.$$

(Since S_n is integer-valued, $\inf_n S_n$ is attained unless it is $-\infty$).

Thus for $0 > a \in \mathbb{Z}$, $P(\inf_n S_n \leq a) = \left(\frac{q}{p}\right)^{-a}$,

So $P(\underbrace{-\inf_n S_n}_{Y} \geq -a) = \left(\frac{q}{p}\right)^{-a}$

$$P(Y \in \{0, 1, 2, \ldots, \infty\}) = 1,$$

so $Y = \sum_{k \in \mathbb{N}} 1_{\{Y \geq k\}}$.

$$\begin{aligned}
\text{So } E(Y) &= \sum_{k \in \mathbb{N}} P(Y \geq k) \\
&= \sum_{k=1}^{\infty} \left(\frac{q}{p}\right)^k \\
&= \frac{q}{p} \frac{1}{1 - \frac{q}{p}} \\
&= \frac{q}{p-q}
\end{aligned}$$

In particular, $E(Y) < \infty$.

<u>Propn</u> Let $0 < b \in \mathbb{Z}$.

Then $E(T_b) = \dfrac{b}{p-q} = \dfrac{b}{E(\underset{=S_1}{\xi_1})}$.

<u>Pf</u> Let $Z_n = S_n - (p-q)n$.

Then $Z_n = \sum_{m \le n} \xi'_m$ where $\xi'_n = \xi_n - (p-q)$.

$\xi'_1, \xi'_2, \xi'_3, \ldots$ are independent, and $E(\xi'_n) = 0$.

Hence, for each n, $E(Z_{T_b \wedge n}) = \overset{0}{\overset{\shortparallel}{E(Z_1)}}\, E(T_b \wedge n) = 0$.

Wald's first Eqn

($T_b \wedge n$ is a stopping time with $E(T_b \wedge n) < \infty$ and (Z_n) is a RW with $E(Z_1) = 0$).

But $Z_{T_b \wedge n} = S_{T_b \wedge n} - (p-q) T_b \wedge n$

So $(p-q) E(T_b \wedge n) = E(S_{T_b \wedge n})$

$$\begin{array}{ccc} (p-q)\, E(T_b \wedge n) & = & E(S_{T_b \wedge n}) \\ \Big\downarrow \text{M.C.T.} & & \Big\downarrow * \\ (p-q)\, E(T_b) & = & E(S_{T_b}) \end{array}$$

$*$ For each $\omega \in \{T_b < \infty\}$,

for each $n \geq T_b(\omega)$,

$S_{T_b \wedge n}(\omega) = S_{T_b}(\omega)$.

So $S_{T_b \wedge n} \longrightarrow S_{T_b} = b$ a.s.

So $S_{T_b \wedge n} \longrightarrow S_{T_b} = b$ a.s.

Also, $\inf_m S_m \leq S_{T_b \wedge n} < b$.

and $0 > E(\inf_m S_m) > -\infty$, $0 < E(b) = b$.

So apply the D.C.T.

So $(p-q)\,E(T_b) = b$,

So $E(T_b) = \dfrac{b}{p-q}$. $\square$

Propn Let $0 < b \in \mathbb{Z}$.

Then $\mathrm{var}(T_b) = b\,\dfrac{1-(p-q)^2}{(p-q)^3}$.

pf Let ξ_k' and Z_n be as before.

Apply Wald's 2nd equation. $\square$

More About Symmetric Simple RW

(Reference: Feller, Vol. 1)

A path is a finite sequence

$(k_0, x_0), \ldots, (k_n, x_n) \in \mathbb{Z} \times \mathbb{Z}$

such that $k_j = k_{j-1} + 1$ and $|x_j - x_{j-1}| = 1$

for each $j = 1, \dots, n$. (usually $k_0 \geq 0$).

Such a path is said to be from (k_0, x_0) to (k_n, x_n) and is said to be of length n. Thus the length of the path is the number of segments $(k_{j-1}, x_{j-1}) \longrightarrow (k_j, x_j)$.

The number of positive steps in such a path is

$$a = \left|\{ j \in \{1, \dots, n\} : x_j - x_{j-1} = 1 \}\right|,$$

and the number of negative steps is

$$b = \left|\{ j \in \{1, \dots, n\} : x_j - x_{j-1} = -1 \}\right|.$$

Clearly $a + b = n$,

and
$$a - b = \sum_{j=1}^{n} (x_j - x_{j-1})$$
$$= x_n - x_0.$$

Hence $a = \dfrac{n + (x_n - x_0)}{2}$, and $b = \dfrac{n - (x_n - x_0)}{2}$

Since a and b are integers ≥ 0,

$$-n \leq x_n - x_0 \leq n$$

and the integers n and $x_n - x_0$ are either both even or both odd.

Conversely, given integers

k_0, n, x_0, and x_n such that

$$-n \leq x_n - x_0 \leq n \qquad \text{(this implies } n \geq 0\text{)}$$

and such that n and $x_n - x_0$ are either both even or both odd, then the set of paths from (k_0, x_0) to (k_0+n, x_n) is in one-to-one correspondence with the set of a-element subsets of $\{1, \dots, n\}$ ($\{1, \dots, n\} = \emptyset$ if $n=0$), where $a = \dfrac{n+(x_n-x_0)}{2}$.

So the number of such paths is $\binom{n}{a}$, which we'll denote by

$N_{n, x_n - x_0}$.

Of course, this is the same as

the number of paths from

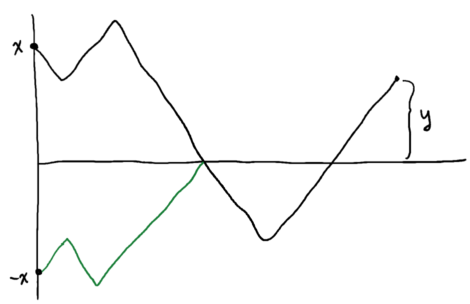
$(0,0)$ to (n, x) where $x = x_n - x_0$.

The Reflection Principle (Desiré André, 1887).

Let x and y be integers > 0.

Then the number of paths from $(0,x)$ to (n,y) that are 0 at some time is equal to the number of paths from $(0,-x)$ to (n,y).

Pf



Let $(0, s_0), (1, s_1), \ldots, (n, s_n)$ be a path from $(0, x)$ to (n, y) such that $s_k = 0$ for some $k \in \{1, \ldots, n-1\}$. Let K be the least such k.

Let $s_k' = \begin{cases} -s_k & \text{if } k < K, \\ s_k & \text{if } k \geq K \end{cases}$ for $k = 0, 1, \ldots, n$.

Then $(0, s_0'), (1, s_1'), \ldots, (n, s_n')$ is a path from $(0, -x)$ to (n, y), because each increment $s_k' - s_{k-1}'$, being either $s_k - s_{k-1}$, $-s_k - (-s_{k-1})$

$s'_k - s'_{k-1}$, being either $\underbrace{s_k - s_{k-1}}_{\pm 1}$, $\underbrace{-s_k - (-s_{k-1})}_{\mp 1}$, or $s_k - (-s_{k-1})$,

and the third case only happens if $k = K+1$, so

$s_k = \pm 1$, and $s_{k-1} = 0$.

The rest of the proof

consists of going back the other way...