Theorem Let $(M_n)_{n\geq 0}$ be a mtgle wrt a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $(H_n)_{n\geq 1}$ be a predictable sequence of real RVs ("Predictable" means for each $n\geq 1$, H_n is $\mathcal{F}_{n-1}$-mble). Suppose for each $n\geq 1$, $H_n(M_n - M_{n-1})$ is integrable. Then $H\cdot M$ is a mtgle.

Proof Remember

$$(H\cdot M)_n = \begin{cases} 0 & \text{if } n=0, \\ \sum_{k=1}^{n} H_k(M_k - M_{k-1}) & \text{if } n\geq 1. \end{cases}$$

Since $H_k(M_k - M_{k-1}) \in L^1$ for each $k\geq 1$,

$(H\cdot M)_n \in L^1$ for each $n\geq 0$.

For each $n\geq 0$, for each $A\in\mathcal{F}_n$,

$$\int_A \left((H\cdot M)_{n+1} - (H\cdot M)_n\right) dP$$

$$= \int_A H_{n+1}(M_{n+1} - M_n)\, dP$$

$$= \int \underbrace{1_A H_{n+1}}\,(M_{n+1} - M_n)\, dP$$

$\mathcal{F}_n$-measurable

$= 0$

because $1_A H_{n+1}$ is $\mathcal{F}_n$-mble and $1_A H_{n+1}(M_{n+1} - M_n) \in L^1$

This is justified by the following lemma:

Lemma Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let $f \in L^1(\mu)$, let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$,

and Suppose for each $B \in \mathcal{B}$, $\int_B f \, d\mu = 0$.

Let $\underbrace{g \in L^0(\mu|_{\mathcal{B}})}_{g \text{ is } \mathcal{B}\text{-mble and } \mathbb{R}\text{- or } \mathbb{C}\text{- valued.}}$ s.t. $fg \in L^1(\mu)$. Then $\int fg \, d\mu = 0$.

Pf: Case 0: if $g = 1_B$ where $B \in \mathcal{B}$, we assumed that $\int fg \, d\mu = 0$.

Case 1: Suppose g is simple wrt $\mathcal{B}$. Then $g = \sum_{k=1}^{n} b_k 1_{B_k}$

for some $n \in \mathbb{N}$, some $b_1, \dots, b_n \in \mathbb{R}$ or $\mathbb{C}$, $B_1, \dots, B_n \in \mathcal{B}$

Then $fg = \sum_{k=1}^{n} b_k f 1_{B_k} \in L^1(\mu)$, and

$$\int fg \, d\mu = \sum_{k=1}^{n} b_k \underbrace{\int f 1_{B_k} \, d\mu}_{0} = 0$$

Case 2: (The general case)

There is a sequence (g_n) of $\mathcal{B}$-simple $\mathbb{R}$- or $\mathbb{C}$-valued fns such that for each x, $g_n(x) \longrightarrow g(x)$ as $n \longrightarrow \infty$,

and for each n, $|g_n(x)| \leq |g(x)|$.

(for each of $(\mathrm{Re}\, g)^{\pm}$, $(\mathrm{Im}\, g)^{\pm}$, take increasing sequences of simple fns & compose/add them).

Then $fg_n \longrightarrow fg$ pointwise as $n \longrightarrow \infty$

and $|fg_n| \leq |fg|$.

By assumption, $\int |fg| \, d\mu < \infty$.

hence $\int fg_n \, d\mu \longrightarrow \int fg \, d\mu$ by DCT.

but $\int fg_n d\mu = 0 \ \forall n$ by case 1,

So $\int fg \, d\mu = 0$.

The Optional Stopping Theorem

Let (M_n) be a mtgle wrt a filtration $(\mathcal{F}_n)$.

Let T be a stopping time. Let $M_n^T = M_{T \wedge n}$ for each n.

(M^T is "M stopped at T"). Then M^T is a mtgle.

Pf $M_{T \wedge n} - \underbrace{M_0}_{\mathcal{F}_0\text{-mble}} = (H \cdot M)_n$ where $H_k = 1_{\{k \leq T\}}$.

(H_k) is predictable because $\{k \leq T\} = \{T \leq k-1\}^c \in \mathcal{F}_{k-1}$

and $0 \leq H_k \leq 1$ so $H_k(M_k - M_{k-1}) \in L^1$. $\square$

Analysis of Asymmetric Simple RW on $\mathbb{Z}$

Let $\xi_1, \xi_2, \xi_3, \ldots$ be independent $\{-1,1\}$-valued RVs.

Let $\frac{1}{2} < p < 1$ and suppose that $\forall k$, $P(\xi_k = 1) = p$,

and $P(\xi_k = -1) = q$, where $q = 1-p$. Let $S_n = \sum_{k \le n} \xi_k$ $\quad (S_0 = 0)$.

(S_n) is called an asymetric simple RW on $\mathbb{Z}$.

define φ on $\mathbb{Z}$ by $\varphi(x) = \left(\frac{q}{p}\right)^x$.

Let $X_k = \varphi(\xi_k)$. Then $X_1, X_2, X_3, \ldots$ are independent

and $E[X_k] = p\left(\frac{q}{p}\right)^1 + q\left(\frac{q}{p}\right)^{-1} = p + q = 1$.

Let $M_n = \prod_{k \le n} X_k = \varphi(S_n) \quad (M_0 = 1)$.

Then (M_n) is a mtgle.

For each $x \in \mathbb{Z}$, let $T_x = \inf\{n : S_n = x\}$.

<u>Propn</u> As $n \longrightarrow \infty$, $S_n \longrightarrow \infty$ a.s.

<u>Pf</u> $\frac{S_n}{n} \longrightarrow E(\xi_1)$ a.s. by the strong

$\qquad\qquad\qquad\quad \shortparallel$

this case was

$\dfrac{\quad}{n} \longrightarrow E(\xi_1)$ a.s. by the Strong law of large numbers.

$\overset{\|}{p-q} > 0$

this case was proved earlier.

So $S_n \longrightarrow \infty$ a.s. $\square$

<u>Corollary</u> Let $1 \le b \in \mathbb{Z}$. Then $T_b < \infty$ a.s.

Pf Let $G = \{\omega \in \Omega : S_n(\omega) \to \infty \text{ as } n \to \infty\}$. Then $P(G) = 1$ by the proposition. $S_0(\omega) = 0$ and $S_n(\omega) - S_{n-1}(\omega) = \pm 1$ for each $n \ge 1$. Since $S_n(\omega) \to \infty$, $S_n(\omega) = b$ for some n and $T_b(\omega) \le n < \infty$. $\square$

<u>Propn</u> Let $a, b \in \mathbb{Z}$ with $a < 0 < b$.

Then
$$P(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$$

<u>pf</u> Let $N = T_a \wedge T_b$. Then $N < \infty$ a.s.

Let $Y_n = \varphi(S_{N \wedge n}) = M_{N \wedge n}$. (Y_n) is a mtgle since N is a stopping time and (M_n) is a mtgle (use the optional Stopping Theorem).

For each $\omega \in \{N < \infty\}$, $Y_n(\omega) \longrightarrow \varphi(S_N(\omega))$.

Since $N < \infty$ a.s., $Y_n \longrightarrow \varphi(S_N)$ a.s.

$E(Y_0) = E(\overset{1}{\overset{\|}{M_0}}) = 1$. Since Y_n is a mtgle, $E(Y_n) = E(Y_0) = 1$.

$a \le S_{N\wedge n} \le b$, so $\varphi(a) \ge Y_{N\wedge n} \ge \varphi(b)$ $\left(\varphi(x) = \left(\frac{q}{p}\right)^x \right.$ is decreasing in x since $0<q<p$).

Hence by DCT, $E(Y_n) \longrightarrow E(\varphi(S_N))$, so $E(\varphi(S_N)) = 1$ since each $E(Y_n) = 1$.

On $\{T_a < T_b\}$, $S_N = a$.

On $\{T_b < T_a\}$, $S_N = b$.

The only way T_a and T_b can be equal is if both are infinite.

So $P(T_a = T_b) \le P(T_b = \infty) = 0$.

Thus $1 = E(\varphi(S_N)) = \varphi(a)\,P(T_a<T_b) + \varphi(b)\,P(T_b<T_a)$

$$1 = P(T_a<T_b) + P(T_b<T_a).$$

So $1 = \varphi(a)\,P(T_a<T_b) + \varphi(b)\left(1 - P(T_a<T_b)\right)$

so $1 - \varphi(b) = \left(\varphi(a) - \varphi(b)\right) P(T_a<T_b)$

so $P(T_a<T_b) = \dfrac{\varphi(b) - \overset{=1}{\varphi(0)}}{}$

$$\text{so } P(T_a < T_b) = \frac{\varphi(b) - \overset{=1}{\varphi(0)}}{\varphi(a) - \varphi(0)} \qquad \square$$

<u>Propn</u> Let $0 > a \in \mathbb{Z}$. $P(T_a < \infty) = \left(\frac{q}{p}\right)^{-a} = \varphi(-a)$

~~Pf~~ let $\omega \in \{T_a < \infty\}$. Then $\sup_n S_{T_a \wedge n}(\omega) = \max\{S_0(\omega), \dots, S_{T_a}(\omega)\} < \infty$.

So $\exists\, b \in \mathbb{Z}$, $b > 0$, such that $T_b(\omega) > T_a(\omega)$.

(just take b bigger than the sup above).

Thus $\{T_a < \infty\} = \uparrow\bigcup_{b \geq 1} \{T_a < T_b\}$, and so as $b \to \infty$,

$$P(T_a < T_b) \longrightarrow P(T_a < \infty).$$

$$\| $$

$$\frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)} \xrightarrow[\text{since } \varphi(b) \to 0]{} \frac{\varphi(0)}{\varphi(a)} = \frac{1}{\varphi(a)} = \left(\frac{q}{p}\right)^{-a}. \qquad \square$$