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$(X, \mathcal{A}, \mu)$ a measure space.

<u>Riesz-Fischer Theorem</u>: $L^2(\mu)$ is complete.

<u>Propn</u> Suppose $f_n \to f$ in $L^2(\mu)$. Then (f_n) has a subsequence (f_{n_k}) s.t. $f_{n_k} \to f$ a.e.

pf $\int |f - f_n|^2 \, d\mu = \|f - f_n\|_2^2 \longrightarrow 0$. Hence $\exists$ natural numbers $n_1 < n_2 < n_3 < \cdots$ such that $\forall k$, $\forall n \geq n_k$, $\int |f - f_n|^2 \, d\mu \leq 2^{-k}$.

In particular, for each k, $\int |f - f_{n_k}|^2 \, d\mu \leq 2^{-k}$.

$$\int \sum_{k=1}^{\infty} |f - f_{n_k}|^2 \, d\mu = \sum_{k=1}^{\infty} \int |f - f_{n_k}|^2 \, d\mu \leq \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty.$$

So $\sum_{k=1}^{\infty} |f - f_{n_k}|^2$ is finite a.e.

Let $X_1 = \left\{ x \in X : \sum_{k \geq 1} |f - f_{n_k}|^2 < \infty \right\}$.

Then $\mu(X \setminus X_1) = 0$, and $\forall x \in X_1$,

$|f(x) - f_{n_k}(x)|^2 \longrightarrow 0$, so $f_{n_k}(x) \longrightarrow f(x)$, so $f_{n_k} \to f$ a.e. $\square$

<u>Corollary of Proof</u>: Let f, f, f $\in L^2(\mu)$. Suppose $\sum \|f_n \ldots\|^2$

Corollary of Proof: Let $f, f_1, f_2, \ldots \in L^2(\mu)$. Suppose $\sum_n \int |f-f_n|^2 d\mu < \infty$,

Then $f_n \longrightarrow f$ a.e.

Propn Suppose $f_n \longrightarrow f$ in $L^2(\mu)$ and $f_n \longrightarrow g$ a.e. Then $f=g$ a.e.

Pf Since $f_n \longrightarrow f$ in $L^2(\mu)$, (f_n) has a subsequence (f_{n_k}) s.t.

$f_{n_k} \longrightarrow f$ a.e. but $f_{n_k} \longrightarrow g$ a.e. too, so $f=g$ a.e.

(since the union of two μ-null sets is μ-null). $\square$

Propn Let (f_n) be an orthogonal sequence in $L^2(\mu)$.

Suppose $\sum_{n=1}^{\infty} \|f_n\|_2^2 < \infty$, and $\sum_{n=1}^{\infty} f_n$ converges a.e. to f.

Then $f \in L^2(\mu)$, $\sum_{n=1}^{\infty} f_n$ converges to f in $L^2(\mu)$, and

$$\sum_{n=1}^{\infty} \|f_n\|_2^2 = \|f\|_2^2.$$

Pf Let $g_n = \sum_{k=1}^{n} f_k$. We have $g_n \longrightarrow f$ a.e., by assumption.

For $m<n$, we have $\|g_m - g_n\|_2^2 = \left\|\sum_{k=m+1}^{n} f_k\right\|_2^2 \overset{(f_n) \text{ is orthogonal}}{=} \sum_{k=m+1}^{n} \|f_k\|_2^2 \xrightarrow[m,n\to\infty]{} 0$

because $\sum_{k=1}^{\infty} \|f_k\|_2^2 < \infty$. Thus the sequence (g_n)

is Cauchy in $L^2(\mu)$. Hence, by the Riesz-Fischer

Theorem, $\exists\, g \in L^2(\mu)$ s.t. $g_n \longrightarrow g$ in $L^2(\mu)$.

But, by assumption, $g_n \longrightarrow f$ a.e. So $g = f$ a.e. by the previous proposition. Hence $f \in L^2(\mu)$ and $g_n \longrightarrow f$ in $L^2(\mu)$,

For each n, $\|g_n\|_2^2 = \left\| \sum_{k=1}^{n} f_k \right\|_2^2 = \sum_{k=1}^{n} \|f_k\|_2^2$.

Since $g_n \longrightarrow f$ in $L^2(\mu)$, $\|g_n\|_2 \longrightarrow \|f\|_2$ because

$$
\begin{array}{ccccc}
-\|f - g_n\|_2 & \leq & \|f\|_2 - \|g_n\|_2 & \leq & \|f - g_n\|_2 \\
\downarrow & & \downarrow \text{squeeze} & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

So $\|g_n\|_2^2 \longrightarrow \|f\|^2$, and so $\sum_{k=1}^{\infty} \|f_k\|_2^2 \longrightarrow \|f\|_2^2$. $\square$

This ties up the loose end in our proof of Wald's 2nd Equation.

Martingales

Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration.

To say $(M_n)_{n \geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)$ means that (M_n) is an $(\mathcal{F}_n)$-adapted sequence of $\int$-ble real RVs such that for each n, for each $A \in \mathcal{F}_n$,

$$E(M_{n+1}\,;A) = E(M_n\,;A)$$

<u>Remark</u> If (M_n) is a mtgle w.r.t. a filtration $(\mathcal{F}_n)$, then for each $n \geq 0$ and for each $k \geq 1$ and for each

$$A \in \mathcal{F}_n, \quad E(M_{n+k}\,;A) = E(M_n\,;A)$$

eg Let (S_n) be a RW in $\mathbb{R}$ wrt a filtration $(\mathcal{F}_n)$.

Suppose $E(|S_1|) < \infty$ and $E(S_1) = 0$.

Then (S_n) is a martingale wrt $(\mathcal{F}_n)$

Pf Let $A \in \mathcal{F}_n$. Then $E(\underbrace{S_{n+1} - S_n}_{X_{n+1} \perp A}\,;A) = E(S_{n+1}-S_n)\cdot P(A) = 0.$ $\square$

eg A Symmetric simple RW on $\mathbb{Z}$ is a mtgle.

eg Let $X_1, X_2, X_3, \ldots$ be integrable real RVs adapted to a filtration $(\mathcal{F}_n)$.

Suppose for each $n \geq 1$, X_n is indp of $\mathcal{F}_{n-1}$.

(a) Suppose $\forall n \geq 1$, $E(X_n) = 0$. Let $S_n = \sum_{k \leq n} X_k$ for $n = 0, 1, 2, \ldots$.

Then (S_n) is a martingale wrt $(\mathcal{F}_n)$.

(b) Suppose instead that $\forall n \geq 1$, $E(X_n) = 1$. Let $M_n = \prod_{k \leq n} X_k$ for $n = 0, 1, 2, \ldots$. Then M_n is a martingale wrt $(\mathcal{F}_n)$.

pf (a) essentially already done, see last example

(b) $X_1, \ldots, X_n$ are independent.

So $E(|M_n|) = E(|X_1| \cdots |X_n|) = E(|X_1|) \cdots E(|X_n|) = 1 < \infty$,

So M_n is integrable. Let $A \in \mathcal{F}_n$.

$$E(M_{n+1}; A) = E(\underbrace{X_{n+1}}_{\substack{\uparrow \\ \text{indp of} \\ \mathcal{F}_n}} \underbrace{M_n 1_A}_{\mathcal{F}_n\text{-mble}}) = E(X_{n+1}) \cdot E(M_n 1_A) = E(M_n; A). \quad \square$$

eg Let (S_n) be an asymmetric simple RW on $\mathbb{Z}$, and let $\xi_n = S_n - S_{n-1}$.

Let $p = P(\xi_1 = 1)$, $q = P(\xi_1 = -1)$. Then $p + q = 1$.

Assume $\frac{1}{2} < p < 1$. Define φ on $\mathbb{Z}$ by $\varphi(x) = \left(\frac{q}{p}\right)^x$.

Then the sequence $(\varphi(S_n))$ is a martingale wrt the filtration $(\mathcal{F}_n = \sigma(S_0, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n))$.

pf $\varphi(S_n) = \prod_{k \leq n} X_k$ where $X_k = \left(\frac{q}{p}\right)^{\xi_k}$.

$$E(X_k) = \frac{q}{p} \cdot P(\xi_k = 1) + \frac{p}{q} \cdot P(\xi_k = -1) = q + p = 1.$$

Martingale Transforms

Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration.

Let $(M_n)_{n \geq 0}$ be a mtgle wrt $(\mathcal{F}_n)$.

Let $(H_n)_{n \geq 1}$ be a predictable process

where for each n, H_n is a bounded real RV.

(Predictable means for each $n \geq 1$, H_n is $\mathcal{F}_{n-1}$-mble)

By defn,

$$(H \circ M)_n = \begin{cases} \sum_{k=1}^{n} H_k (M_k - M_{k-1}) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$