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Monday, April 1, 2019 14:19

Poisson Random Measures

Let $(E, \mathcal{E})$ be a mble space.

Let λ be a measure on $\mathcal{E}$.

Suppose $0 < \lambda(E) < \infty$

Let $p = \frac{1}{\lambda(E)} \lambda$. p is a probability measure on $\mathcal{E}$.

Let $N, X_1, X_2, X_3, \ldots$ be independent RVs such that N takes values in $\{0, 1, 2, \ldots\}$ and is Poisson distributed with parameter $\lambda(E)$, and for each j, X_j takes values in E and has distribution p. (So $P(X_j \in A) = p(A)$ for each $A \in \mathcal{E}$).

For each $\omega \in \Omega$ and for each $A \in \mathcal{E}$, let

$N(\omega, A) = \#\{ j \leq N(\omega) : X_j(\omega) \in A \}$.

Note that for each $\omega \in \Omega$, the function

$A \longmapsto N(\omega, A)$ is a finite measure on $\mathcal{E}$

taking values in the set $\{0, 1, 2, \ldots, N(\omega)\}$.

Also, for each $A \in \mathcal{E}$, the function

$\omega \longmapsto N(\omega, A)$ is measurable.

Both of these assertions follow from the equalities

$$N(\omega, A) = \sum_{j=1}^{N(\omega)} \delta_{X_j(\omega)}(A)$$

(Notation: $\delta_x(A) = 1_A(x)$, the unit point mass at x)

$$= \sum_{j=1}^{\infty} 1_{[j,\infty)}(N(\omega)) \cdot \delta_{X_j(\omega)}(A)$$

$$= \sum_{j=1}^{\infty} 1_{[j,\infty)}(N(\omega)) \cdot 1_A(X_j(\omega))$$

For each $A \in \mathcal{E}$, let us also write $N(A)$ for the function $\omega \longmapsto N(\omega, A)$. The symbol N can stand for any one of three different functions:

(1) The original Poisson Distributed RV N with parameter $\lambda(E)$.

(2) The function $(\omega, A) \longrightarrow N(\omega, A)$.

(3) The function $A \longmapsto N(A)$.

Note that the N in (1) is $N(A)$ with $A = E$ in (3).

Now let $A_1, \ldots, A_r$ be disjoint sets belonging to $\mathcal{E}$ with $A_1 \cup \cdots \cup A_r = E$. We claim that the RVs

$N(A_1), \dots, N(A_r)$ are independent and Poisson distributed with parameters $\lambda(A_1), \dots, \lambda(A_r)$ respectively.

Let $n_1, \dots, n_r \in \{0, 1, 2, \dots\}$. We wish to show that

$$P(N(A_1) = n_1, \dots, N(A_r) = n_r) = \prod_{\ell=1}^{r} \frac{(\lambda(A_\ell))^{n_\ell}}{n_\ell!} e^{-\lambda(A_\ell)} .$$

Let $n = n_1 + \dots + n_r$. For $\ell = 1, \dots, r$, let $Z_\ell = \#\{j \le n : X_j \in A_\ell\}$.

(This means that for each ω, $Z_\ell(\omega) = \#\{j \le n : X_j(\omega) \in A_\ell\}$).

Observe that for $\ell = 1, \dots, r$, we have

$$N(A_\ell) = Z_\ell \quad \text{on} \quad \{N = n\} .$$

Also for $\ell = 1, \dots, r$, Z_ℓ is mble w.r.t. $\sigma(X_1, \dots, X_n)$

because $Z_\ell = \sum_{j=1}^{n} 1_{A_\ell}(X_j)$.

Thus $(Z_1, \dots, Z_n)$ and N are independent.

Now
$$\begin{aligned} P(N(A_1) = n_1, \dots, N(A_r) = n_r) &= P(N(A_1) = n_1, \dots, N(A_r) = n_r, N = n) \\ &= P(Z_1 = n_1, \dots, Z_r = n_r, N = n) \\ &= P(Z_1 = n_1, \dots, Z_r = n_r) \cdot P(N = n) . \end{aligned}$$

Note that for $\ell = 1, \dots, r$ we have

$$Z_\ell = |\{j \le n : Y_j = \ell\}|$$

where for $j = 1, \dots, n$,

$$Y_j = \begin{cases} 1 & \text{on } \{X_j \in A_1\}, \\ \vdots & \\ r & \text{on } \{X_j \in A_r\}. \end{cases}$$

Of course $Y_1, \dots, Y_n$ are independent since $X_1, \dots, X_n$ are independent. They take values in $\{1, \dots, r\}$ and satisfy $P(Y_j = \ell) = P(X_j \in A_\ell) = p(A_\ell)$

Thus $(Z_1, \dots, Z_r)$ has a multinomial distribution with parameters $n, r, p(A_1), \dots, p(A_r)$, so

$$P(Z_1 = n_1, \dots, Z_r = n_r) = \frac{n!}{n_1! \cdots n_r!} p(A_1)^{n_1} \cdots p(A_r)^{n_r}$$

$$= \frac{n!}{n_1! \cdots n_r!} \cdot \frac{\lambda(A_1)^{n_1}}{\lambda(E)^{n_1}} \cdots \frac{\lambda(A_r)^{n_r}}{\lambda(E)^{n_r}} .$$

Also, $P(N = n) = e^{-\lambda(E)} \dfrac{\lambda(E)^n}{n!}$

$$= e^{-(\lambda(A_1) + \cdots + \lambda(A_r))} \cdot \frac{\lambda(E)^{n_1 + \cdots + n_r}}{n!}$$

Hence $P(N(A_1) = n_1, \dots, N(A_r) = r) = e^{-\lambda(A_1)} \dfrac{\lambda(A_1)^{n_1}}{n!} \cdot \dots \cdot e^{-\lambda(A_r)} \dfrac{\lambda(A_r)^{n_r}}{n_r!}$, as claimed.

The family of RVs $(N(A))_{A\in\mathcal{E}}$ is an example of what is called a Poisson random measure with parameter measure λ.

Now let us assume, in addition, that $(E,\mathcal{E})$ is countably separated. Note that then $\forall x\in E$, we have $\{x\}\in\mathcal{E}$, because if (H_j) is a sequence in $\mathcal{E}$ which separates points in E and if $G_j=\begin{cases}H_j & \text{if } x\in H_j,\\ E\setminus H_j & \text{otherwise,}\end{cases}$

Then $\bigcap_{j=1}^{\infty} G_j=\{x\}$.

Let us also assume that for each $x\in E$, $\lambda(\{x\})=0$.

Under these two additional assumptions, we shall show that the poisson random measure we have just constructed can be modified on a set of probability zero so that it will be the counting measure for a certain random subset of E.

Let $\Delta = \{(x,x) : x \in E\}$. Then $\Delta \in \mathcal{E} \otimes \mathcal{E}$, because if (H_j) is a sequence in $\mathcal{E}$ which separates points in E, then

$$\Delta = E \times E \setminus \bigcup_{j=1}^{\infty} \left[(H_j \times (E \setminus H_j)) \cup ((E \setminus H_j) \times H_j) \right].$$

for all $j_1, j_2 \in \{1,2,3,\dots\}$, if $j_1 \neq j_2$ then

$$\begin{aligned} P(X_{j_1} = X_{j_2}) &= P((X_{j_1}, X_{j_2}) \in \Delta) \\ &= \int_E \left[\int_E 1_\Delta(x,y)\, p(dy) \right] p(dx) \\ &= \int_E \left[\int_E 1_{\{x\}}(y)\, p(dy) \right] p(dx) \\ &= \int_E p(\{x\})\, p(dx) \\ &= \int_E \overset{\downarrow \text{ by assumption}}{0}\, p(dx) \\ &= 0 \end{aligned}$$

Let $\Omega_1 = \{\omega \in \Omega : X_{j_1}(\omega) \neq X_{j_2}(\omega) \text{ for all } j_1 \neq j_2\}$

$$= \bigcap_{j_1, j_2} \{X_{j_1} \neq X_{j_2}\}.$$

Then $P(\Omega_1) = 1$.

Redefine the original N to be 0 on $\Omega \setminus \Omega_1$

and redefine $N(\omega, A)$, and $N(A)$ accordingly.

Then $(N(A))_{A \in \mathcal{E}}$ is still a poisson random measure with mean measure λ, but now for each $\omega \in \Omega$ and for each $A \in \mathcal{E}$, we have

$$N(\omega, A) = |\Pi(\omega) \cap A| \quad \text{where}$$

$$\Pi(\omega) = \{X_j(\omega) : j \leq N(\omega)\}$$

The random set Π is an example of a Poisson random set with mean measure λ. $\square$

<u>Theorem</u> Let $\lambda \in (0, \infty)$. Then there is a poisson Process $(N_t)_{0 \leq t \leq 1}$ with rate λ.

<u>Proof</u> Let $\mu = \lambda \cdot m$ where m is the Borel-Lebesgue measure on $[0,1]$. Let Π be a poisson random set in $[0,1]$ with mean measure μ. We may take $N_t(\omega) = |\Pi(\omega) \cap (0, t]|$. $\square$