Compare $\log(n!)$ with $\dfrac{L_n + V_n}{2}$

Where $L_n = n\log n - n$

$V_n = (n+1)\log n - n$

$$\frac{L_n + V_n}{2} = (n+\tfrac{1}{2})\log n - n$$

let $d_n = \log(n!) - \dfrac{L_n + V_n}{2}$

then $d_n = \log(n!) - (n+\tfrac{1}{2})\log n + n$

hence
$$\begin{aligned} d_n - d_{n+1} &= \log(n!) - (n+\tfrac{1}{2})\log n + n \\ &\quad - \log((n+1)!) + (n+\tfrac{3}{2})\log(n+1) - (n+1) \\ &= -\log(n+1) + (n+\tfrac{1}{2})\left(\log(n+1) - \log n\right) + \log(n+1) - 1 \\ &= (n+\tfrac{1}{2})\log\left(\frac{n+1}{n}\right) - 1. \end{aligned}$$

Now
$$\frac{n+1}{n} = \frac{2n+2}{2n} = \frac{(2n+1)+1}{(2n+1)-1} = \frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}.$$

For $-1 < x < 1$, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

So (integrating), $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

So $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots$

so $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$.

Hence
$$
\begin{aligned}
d_n - d_{n+1} &= \left(n+\tfrac{1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1 \\
&= \tfrac{1}{2}(2n+1)\log\left(\frac{1+\frac{1}{2n+1}}{1+\frac{1}{2n+1}}\right) - 1 \\
&= \frac{1}{3}\left(\frac{1}{2n+1}\right)^2 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^4 + \frac{1}{7}\left(\frac{1}{2n+1}\right)^6 + \cdots
\end{aligned}
$$

Thus $d_n - d_{n+1} > 0$, so (d_n) is strictly decreasing.

Also,
$$
\begin{aligned}
d_n - d_{n+1} &< \frac{1}{3}\left(\frac{1}{2n+1}\right)^2 + \frac{1}{3}\left(\frac{1}{2n+1}\right)^4 + \frac{1}{3}\left(\frac{1}{2n+1}\right)^6 + \cdots \\
&= \frac{1}{3}\left(\frac{1}{2n+1}\right)^2\left[\frac{1}{1-\left(\frac{1}{2n+1}\right)^2}\right] \\
&= \frac{1}{3}\,\frac{1}{(2n+1)^2-1} \\
&= \frac{1}{3}\,\frac{1}{4n^2+4n} \\
&= \frac{1}{12}\left(\frac{1}{n^2+n}\right)
\end{aligned}
$$

$$12\,(\,n^2+n\,)$$

$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{12n} - \frac{1}{12(n+1)} \quad)$$

So $$d_n - \frac{1}{12n} > d_{n+1} - \frac{1}{12(n+1)}$$

So the sequence $\left(d_n - \frac{1}{12n}\right)$ is strictly increasing.

So $d_1 - \frac{1}{12} < d_n - \frac{1}{12n} < d_n < d_1$

Hence there exists $C \in \mathbb{R}$ such that

$$d_n \downarrow C \quad \text{and} \quad d_n - \frac{1}{12n} \uparrow C$$

as $n \longrightarrow \infty$.

Recalling the defn of d_n, we see that

$$\log(n!) - \left(n + \tfrac{1}{2}\right)\log n + n \downdownarrows C,$$

so $$\log(n!) - \left[C + \left(n+\tfrac{1}{2}\right)\log n - n\right] \downdownarrows 0.$$

Taking exponentials, this means

$$n! \quad ..$$

$$\frac{\quad}{e^{c}\, n^{n+\frac{1}{2}}\, e^{-n}} \downarrow\downarrow 1$$

In particular, $n! \sim e^{c}\, n^{n+\frac{1}{2}}\, e^{-n}$.

Also, $e^{c}\, n^{n+\frac{1}{2}}\, e^{-n} < n!$.

Now $d_n - \frac{1}{12n} \uparrow c$, so $d_n - \frac{1}{12n} < c$.

So $\log(n!) < c + (n+\frac{1}{2})\log n - n + \frac{1}{12n}$

so $n! < e^{c}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n}}$

Now let's prove a sharper lower bound for $n!$.

We bound the series for $d_n - d_{n+1}$ below by its first term,

$$d_n - d_{n+1} > \frac{1}{3}\left(\frac{1}{2n+1}\right)^2$$

$$= \frac{1}{3}\left(\frac{1}{4n^2+4n+1}\right)$$

$$= \frac{1}{12n^2+12n+3}$$

$$> \frac{1}{12n+1} - \frac{1}{12(n+1)+1}$$

because $$\frac{1}{12n+1} - \frac{1}{12(n+1)+1} = \frac{12n+13 - 12n+1}{(12n+1)(12n+13)}$$

$|$

$$= \frac{}{(n+\frac{1}{2})(12n+13)}$$

$$= \frac{1}{12n^2 + 14n + \frac{13}{12}}$$

$$= \frac{1}{12n^2 + 12n + 3 + (2n + \frac{13}{12} - 3)}$$

$$< \frac{1}{12n^2 + 12n + 3} \qquad \forall n \geq 1 .$$

Thus $d_n - \dfrac{1}{12n+1} > d_{n+1} - \dfrac{1}{12(n+1)+1}$.

Thus $d_n - \dfrac{1}{12n+1} \downarrow\downarrow C$.

In particular, $d_n - \dfrac{1}{12n+1} > C$, so

$\log(n!) > C + (n+\frac{1}{2}) \log n - n + \dfrac{1}{12n+1}$, so

$$n! > e^C n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} .$$

<u>Remarks</u>

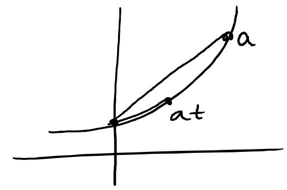
(a) $n!$ exceeds $\underbrace{e^C}_{e^C=\sqrt{2\pi}} n^{n+\frac{1}{2}} e^{-n}$ by less than $\dfrac{9}{n}$ % .

(b) $e^C n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}$ exceeds $n!$ by less than $\dfrac{0.7}{n^2}$ %

pf $e^0 = 1$, $x \longmapsto e^x$ is convex, so if $a > 0$ and $0 < t < 1$,

then $e^{at} - 1$, $e^a - 1$ . .

then $\frac{e^{at}-1}{at} \le \frac{e^{a}-1}{a}$.



So $e^{at} \le 1 + (e^{a}-1)t$, and

hence, for $n \ge 1$, $e^{\frac{1}{12n}} \le 1 + \frac{e^{\frac{1}{12}}-1}{n} < 1 + \frac{0.09}{n}$.

$$e^{\frac{1}{12n} - \frac{1}{12(n+1)}} < e^{\frac{1}{144n^2}} \le 1 + \frac{e^{\frac{1}{144}}-1}{n^2} < 1 + \frac{0.007}{n^2}.$$

The DeMoivre-Laplace Theorem

1718, 1778, 1756   1812

Let $0 < p < 1$, and let $q = 1-p$. Let $X_1, X_2, X_3, \ldots$ be independent Bernoulli (p). So $P(X_j = 1) = p$ and $P(X_j = 0) = q$.

Then $E(X_j) = p$, and $Var(X_j) = pq$.

Let $S_n = X_1 + \cdots + X_n$. Then $E(S_n) = np$ and $Var(S_n) = npq$.

Let $S_n^* = \frac{S_n - np}{\sqrt{npq}}$. The DeMoivre-Laplace theorem states that for $-\infty < a < b < \infty$, $\lim_{n\to\infty} P(a < S_n^* < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

This is the earliest form of the Central Limit Theorem.