$$P\left(\frac{S_n}{n} \geq p+\varepsilon\right) \leq e^{-\lambda\varepsilon n}\left[p\left(\lambda q + e^{\lambda^2 q^2}\right) + q\left(-\lambda p + e^{\lambda^2 p^2}\right)\right]^n$$

$$= e^{-\lambda\varepsilon n}\left[pe^{\lambda^2 q^2} + qe^{\lambda^2 p^2}\right]^n$$

$$\leq e^{-\lambda\varepsilon n}\left[pe^{\lambda^2} + qe^{\lambda^2}\right]^n$$

$$= e^{(\lambda^2 - \lambda\varepsilon)n}$$

$$= e^{\left[(\lambda - \frac{1}{2}\varepsilon)^2 - \frac{1}{4}\varepsilon^2\right]n}$$

This holds $\forall \lambda > 0$. in particular,

if $\lambda = \frac{1}{2}\varepsilon$, we get

$$P\left(\frac{S_n}{n} \geq p+\varepsilon\right) \leq e^{-\frac{1}{4}\varepsilon^2 n}. \qquad \square$$

<u>Remark</u>: $\forall \varepsilon > 0$,

$$P\left(\frac{S_n}{n} \leq p-\varepsilon\right) = P\left(1 - \frac{S_n}{n} \geq 1 - p + \varepsilon\right)$$

$$= P\left(\frac{(1-X_1) + \cdots + (1-X_n)}{n} \geq q + \varepsilon\right)$$

$$\leq e^{-\frac{1}{4}\varepsilon^2 n}$$

Hence, $P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq 2e^{-\frac{1}{4}\varepsilon^2 n}$

In particular, $\sum_{n=1}^{\infty} P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq \sum_{n=1}^{\infty} 2e^{-\frac{1}{4}\varepsilon^2 n} < \infty$

Hence, by the first Borel-Cantelli lemma, $\sum_{n=1}^{\infty} P(G_n(\varepsilon)^c) < \infty$

where $G_n(\varepsilon) = \{\omega \in \Omega : |\frac{S_n(\omega)}{n} - p| < \varepsilon\}$.

$$P(\underbrace{\omega \in \Omega : \omega \in G_n(\varepsilon)^c \text{ for infinitely many } n}_{B(\varepsilon)}) = 0$$

Let $G(\varepsilon) = \{\omega \in \Omega : |\frac{S_n(\omega)}{n} - p| \geq \varepsilon \text{ for just finitely many } n\}$.
$= B(\varepsilon)^c$

Then $P(G(\varepsilon)) = 1$.

For each $\omega \in G(\varepsilon)$, there exists $N = N(\omega)$

such that for each $n \geq N$,

$$\left| \frac{S_n(\omega)}{n} - p \right| < \varepsilon.$$

Let $G = \bigcap_{\varepsilon > 0} G(\varepsilon)$. for all $\varepsilon_1, \varepsilon_2 > 0$,

if $\varepsilon_1 > \varepsilon_2$ then $G(\varepsilon_1) \supseteq G(\varepsilon_2)$.

So $G = \bigcap_{n \in \mathbb{N}} G(\frac{1}{n}) \in \mathcal{F}$

$\cancel{so}$ $P(G) = \lim_{n \to \infty} G(\frac{1}{n}) = 1$ countable sub-additivity.

For each $\omega \in G$, for each $\varepsilon > 0$, we have

$\omega \in G(\varepsilon)$, so there exists N s.t.

for each $n \geqslant N$, $\left| \frac{S_n(\omega)}{n} - p \right| < \varepsilon$.

$$\text{So } \lim_{n \to \infty} \frac{S_n(\omega)}{n} = p \quad \forall\, \omega \in G.$$

Thus $\frac{S_n}{n} \longrightarrow p$ almost surely.

General Principle:

Let $(X, \mathcal{a}, \mu)$ be a measure space.

Let $f, f_1, f_2, f_3, \ldots : X \longrightarrow \mathbb{R}$ be measurable functions.

Suppose $\forall\, \varepsilon > 0$, $\displaystyle\sum_{n=1}^{\infty} \mu(|f - f_n| \geqslant \varepsilon) < \infty$

Then $f_n \longrightarrow f$ μ-a.e.

Lemma (for Bernstein's Inequality):

For each $x \in \mathbb{R}$, $e^x \leq x + e^{x^2}$.

Pf Define f, g on $\mathbb{R}$ by $f(x) = e^{x^2} - e^x$ and $g(x) = -x$.

$$f(x) = \left(1 + x^2 + \cdots\right) - \left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$= -x + \frac{x^2}{2} + \cdots$$

so $f(0) = 0$ and $f'(0) = -1$.

Thus g is the tangent line to f at $x=0$.

Hence it suffices to show that f is convex

$$f'(x) = 2xe^{x^2} - e^x, \quad \text{and} \quad f''(x) = (4x^2+2)\,e^{x^2} - e^x$$
$$= \left(4x^2 + 2 - e^{x-x^2}\right) e^{x^2}$$

Now $4x^2+2 \geq 2$ while $e^{x-x^2} = e^{\frac{1}{4} - (\frac{1}{2}-x)^2} \leq e^{1/4} = 1.2840254\ldots < 2$

$$\left(e^{1/4} \leq 4^{1/4} = 2^{1/2} = 1.41\ldots \right)$$

Hence $f''(x) > 0$ for all $x \in \mathbb{R}$, so

f is convex as desired $\square$

Notation: $a_n \sim b_n$ as $n \longrightarrow \infty$ means $\dfrac{a_n}{b_n} \longrightarrow 1$ as $n \longrightarrow \infty$.

Stirling's Formula: $n! \sim \underbrace{\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}}_{= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$ as $n \longrightarrow \infty$.

In fact, for each $n \in \mathbb{N}$, $\underline{\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n}}}$

Moreover, the following sharper lower bound holds:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n+1}} < n!$$

~~Pf~~ for now, we'll prove these statements with $\sqrt{2\pi}$ replaced by e^c, where C is a suitable constant.

Later we'll give a probabilistic proof that $e^c = \sqrt{2\pi}$.

Since $x \longmapsto \log x$ is increasing for $0 < x < \infty$, we have

$$\int_{k-1}^{k} \log x \, dx < \log k < \int_{k}^{k+1} \log x \, dx$$

for $k = 1, 2, 3, \ldots$

Summing over k from 1 to n, we get

$$\int_0^n \log x \, dx < \log(n!) < \int_1^{n+1} \log x \, dx$$

Evaluating the integrals, we see that

$$L_n < \log(n!) < U_n$$

where $L_n = \int_0^n \log x \, dx = n \log n - n$

$$U_n = \int_1^n \log x \, dx = (n+1) \log(n+1) - n$$

This already shows that

$$n^n e^{-n} < n! < (n+1)^{(n+1)} e^{-n}$$

But we want to do better...