X, Y ind $\Longrightarrow$ uncorrelated

$\not\Longleftarrow$:

$\Omega = [0,1]$

$X(\omega) = \cos(2\pi\omega)$

$Y(\omega) = \sin(2\pi\omega)$

X & Y are uncorrelated:

$E(XY) = 0.$

but X & Y are not independent.

<u>Propn</u>: Let $X_1, \dots, X_n \in L^1$ with $X_i X_j \in L^1 \quad \forall\, i \neq j.$

Let $X = X_1 + \cdots + X_n$. Then

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

pf $\mathrm{Var}(X) = E((X^\circ)^2) = E\Big(\underbrace{\sum_i (X_i^\circ)^2}_{\geq 0} + 2 \underbrace{\sum_{i<j} X_i^\circ X_j^\circ}_{\in L^1}\Big). \quad \square$

<u>Corollary</u>: if X_i's are pairwise uncorrelated, then

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i).$$

The Binomial Distribution

Let $0 < p < 1$ and let $q = 1 - p$.

Let $X_i = 1_{A_i}$ where $A_1, A_2, A_3, \ldots$

are independent events with $P(A_i) = p$.

Let $S_n = X_1 + \cdots + X_n$, so $\forall \omega \in \Omega$,

$$S_n(\omega) = |\{ i \le n : \omega \in A_i \}|$$

for $k = 0, 1, \ldots, n$, $P(S_n = k) = \binom{n}{k} p^k q^{n-k}$

$$E(S_n) = \sum_{i=1}^{n} E(X_i) = np$$

$$Var(S_n) = \sum_{i=1}^{n} Var(X_i) = n\left(p(1-p)^2 + (1-p)p^2\right) = np(1-p)$$

The law of large numbers for Bernoulli trials

As $n \to \infty$, we expect that $\frac{S_n}{n} \longrightarrow p$ in some sense

The L^2 Weak Law of Large numbers:

Let $X_1, X_2, X_3, \ldots \in L^2$ be pairwise uncorrelated.

Assume that $E(X_j) = \bar{x}$ for all j and $\sup_j Var(X_j) = M < \infty$

Let $S_n = X_1 + \cdots + X_n$. Then $E\left[\left(\bar{x} - \frac{S_n}{n}\right)^2\right] \to 0$ as $n \to \infty$.

pf $E\left[\left(\bar{x} - \frac{S_n}{n}\right)^2\right] = Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2}\sum_{j=1}^{n} Var(X_j) \le \frac{1}{n^2} \cdot nM = \frac{M}{n} \to 0.$

Remark: To show that $P(\frac{S_n}{n} \to \bar{x}) = 1$ takes more work.

Example (The typewriter sequence)

Let $B_0 = [0,1)$

$B_1 = [0, \frac{1}{2})$, $B_2 = [\frac{1}{2}, 1)$

$B_3 = [0, \frac{1}{4})$, $B_4 = [\frac{1}{4}, \frac{1}{2})$, $B_5 = [\frac{1}{2}, \frac{3}{4})$, $B_6 = [\frac{3}{4}, 1)$.

Let $\Omega = [0,1)$, P = Lebesgue measure

$Y_n = 1_{B_n}$. Then $E(Y_n^2) \longrightarrow 0$ as $n \to \infty$.

For each $\omega \in \Omega$, $\limsup_{n\to\infty} Y_n(\omega) = 1$.

$\liminf_{n\to\infty} Y_n(\omega) = 0.$

So $P(\lim_{n\to\infty} Y_n \text{ exists}) = 0.$

Bernstein's Inequality Let $0 < p < 1$.

Bernstein's Inequality Let $0 < p < 1$.

Let $A_1, A_2, A_3, \dots$ be independent events with $P(A_i) = p$ $\forall i$. Let $X_i = 1_{A_i}$, $S_n = X_1 + \dots + X_n$, let $\varepsilon > 0$, $n \in \mathbb{N}$. Then $P\left(\frac{S_n}{n} \geq p + \varepsilon\right) \leq \exp\left(-\frac{\varepsilon^2}{4} n\right)$

pf Let $m = \lceil n(p+\varepsilon) \rceil$, $q = 1-p$, and $\lambda > 0$.

Then $P\left(\frac{S_n}{n} \geq p+\varepsilon\right) = P(S_n \geq n(p+\varepsilon)) = \sum_{k=m}^{n} P(S_n = k) = \sum_{k=m}^{n} \binom{n}{k} p^k q^{n-k}$

$$\leq \sum_{k=m}^{n} e^{\lambda[k - n(p+\varepsilon)]} \binom{n}{k} p^k q^{n-k}$$

$$= e^{-\lambda \varepsilon n} \sum_{k=m}^{n} e^{\lambda(pk + qk - pn)} \binom{n}{k} p^k q^{n-k}$$

$$= e^{-\lambda \varepsilon n} \sum_{k=m}^{n} \binom{n}{k} (pe^{\lambda q})^k (qe^{-\lambda p})^{n-k}$$

$$\leq e^{-\lambda \varepsilon n} \sum_{k=0}^{n} \binom{n}{k} (pe^{\lambda q})^k (qe^{-\lambda p})^{n-k}$$

$$= e^{-\lambda \varepsilon n} (pe^{\lambda q} + qe^{-\lambda p})^n.$$

Now $\forall x \in \mathbb{R}$, $e^x \leq x + e^{x^2}$ (See the lemma which follows)

to be continued...