Lemma Let (A_n) be a seq. of events.

Suppose $\forall \omega$, $\exists n$ s.t. $\forall n' \geq n$, we have $\omega \in A_{n'}$.

Then $P(A_n^c) \longrightarrow 0$.

Pf Let $B_n = \bigcap_{n' \geq n} A_{n'}$. Then $B_1 \subseteq B_2 \subseteq \cdots$

Hence $P(B_n) \uparrow P(\bigcup_n B_n)$. But $\forall \omega$, $\exists n$ s.t. $\omega \in B_n$.

So $\bigcup_n B_n = \Omega$. Thus $P(B_n) \uparrow 1$.

Now $\forall n$, $P(A_n) \geq P(B_n)$, so $P(A_n) \longrightarrow 1$

and so $P(A_n^c) \longrightarrow 0$. $\square$

Corollary: Let $(\mathbb{X}, \mathcal{A})$ be a ctbly separated mble space.

Let $X, X_1, X_2, \ldots$ be RVs in $\mathbb{X}$.

Suppose $\forall \omega$, there exists n such that

for each $n' \geq n$, we have $X_{n'}(\omega) = X(\omega)$.

then $P(X_n \neq X) \longrightarrow 0$.

Pf Apply lemma w/ $A_n = \{X_n = X\}$.

Defn Let $(\mathbb{X}, \mathcal{A})$ be a mble space.

To say $(\mathbb{X}, \mathcal{A})$ is <u>countably separated</u> means that there is a sequence (A_n) in $\mathcal{A}$ such that for all $\omega, \omega' \in \mathbb{X}$, if $\omega \neq \omega'$ then for some n, $1_{A_n}(\omega) \neq 1_{A_n}(\omega')$.

eg Let $\mathcal{B} = \text{Borel}(\mathbb{R})$. Then $(\mathbb{R}, \mathcal{B})$ is ctbly separated.

Pf Let (r_n) be an enumeration of $\mathbb{Q}$, and let $A_n = (-\infty, r_n)$.

eg $(\mathbb{R}^d, \text{Borel}(\mathbb{R}^d))$ is also ctbly separated

eg Let $\mathbb{X}$ be a separable metric space. Then $(\mathbb{X}, \text{Borel}(\mathbb{X}))$ is ctbly separated

eg Perhaps $(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathcal{P}(\mathbb{R})))$ is not ctbly separated

<u>Theorem</u> Let $(Y, \mathcal{B})$ be a mble space. Then TFAE:

(a) $(Y, \mathcal{B})$ is ctbly separated

(b) For each mble space $(X, \mathcal{A})$ and for all mble fns $f, g : X \longrightarrow Y$, $\{f \neq g\} \in \mathcal{A}$.

(c) For each mble space $(X, \mathcal{A})$, for each mble fn $f : X \longrightarrow Y$, $f \in \mathcal{A} \otimes \mathcal{B}$.

(d) $id_Y \in \mathcal{B} \otimes \mathcal{B}$

(e) There is a mble space $(X, \mathcal{A})$ and a mble map $f : X \overset{\text{onto}}{\to} Y$ s.t. $f \in \mathcal{A} \otimes \mathcal{B}$.

(e') There is a mble space $(X, \mathcal{A})$ and a map $f : X \xrightarrow{\text{onto}} Y$ s.t. $f \in \mathcal{A} \otimes \mathcal{B}$.

Remark: To show (e') $\Longrightarrow$ (a), use the fact that for each $C \in \sigma(\mathcal{H})$, there exists $\mathcal{H}_0$ ctble $\subseteq \mathcal{H}$ such that $C \in \sigma(\mathcal{H}_0)$.

Theorem Let $(\mathbb{X}, \mathcal{A})$ be a mble space. Then TFAE:

(a) $(\mathbb{X}, \mathcal{A})$ is ctbly sep'd

(b) There is a one-to-one mble fn $f : X \longrightarrow \mathbb{R}$.

pf (b) $\Longrightarrow$ (a) obvious (take preimages of $(-\infty, r_n)$)

(a) $\Longrightarrow$ (b) Let (A_n) be as in defn of ctbly sep'd.

Notice that $\sigma = (s_1, s_2, s_3, \dots) \xmapsto{h} \sum_{n=1}^{\infty} \frac{2 s_n}{3^n}$

is a one-to-one map from $\Sigma = \{0,1\}^{\mathbb{N}}$ into $\mathbb{R}$.

let $f = \sum_{n=1}^{\infty} \frac{2 1_{A_n}}{3^n}$. Then f is mble and one-to-one.

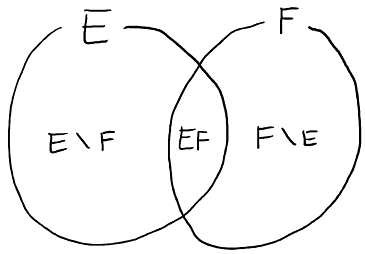
$\left(f = h \circ g \text{ where } g : X \to \Sigma \text{ by } g(x) = (1_{A_n}(x))_{n \in \mathbb{N}} \right)$. $\square$

Notation Let E and F be sets. Then
$$E \Delta F = (E \cup F) \setminus (E \cap F) = (E \setminus F) \cup (F \setminus E) = \{ 1_E \neq 1_F \}.$$

Lemma Suppose E & F are events,

Then $|P(E) - P(F)| \le P(E \Delta F)$.

Pf



$E = EF \sqcup (E \setminus F)$

$F = EF \sqcup (F \setminus E)$

So $|P(E) - P(F)| = |P(EF) + P(E \setminus F) - P(EF) - P(F \setminus E)|$

$\le |P(E \setminus F)| + |P(F \setminus E)|$

$= P(E \setminus F) + P(F \setminus E)$

$= P(E \Delta F)$

<u>Lemma</u> Let $(\mathfrak{X}, \mathcal{A})$ be a ctbly sep'd mble space. Let $X, X_1, X_2, \ldots$ be RVs in $\mathfrak{X}$. Suppose $P(X_n \ne X) \longrightarrow 0$. Then for each $A \in \mathcal{A}$, $P(X_n \in A) \longrightarrow P(X \in A)$.

<u>Pf</u> Let $A \in \mathcal{A}$. Let $F_n = \{X_n \in A\}$, $F = \{X \in A\}$. For each ω, if $\omega \in F \setminus F_n$, then $X(\omega) \in A$, and $X_n(\omega) \notin A$. So $X_n(\omega) \ne X(\omega)$.

Thus $F \setminus F_n \subseteq \{X_n \ne X\}$. Similarly, $F_n \setminus F \subseteq \{X_n \ne X\}$.

Thus $F_n \Delta F \subseteq \{X_n \ne X\}$. Hence $0 \le P(F_n \Delta F) \le P(X_n \ne X) \longrightarrow 0$,

So $P(F_n \Delta F) \longrightarrow 0$. But $|P(F_n) - P(F)| \le P(F_n \Delta F)$,

So $P(F_n) \longrightarrow P(F)$. $\square$

<u>eg</u> In the proof of Thm 1 on the Poisson process, we saw that $\forall \omega$, $\exists n$ s.t. $\forall n' \ge n$, we have $X_{n'}(\omega) = N_t(\omega)$.

Hence $P(X_n \neq N_t) \longrightarrow 0$. Hence $\forall A \subseteq \{0,1,2,\dots\}$, we have $P(X_n \in A) \longrightarrow P(N_t \in A)$. Take $A = \{k\}$. Then we get $P(X_n = k) \longrightarrow P(N_t = k)$.

Now $law(X_n)$ is binomial with parameters n and p_n, where $np_n \longrightarrow \lambda_t \in [0,\infty)$.

Hence $P(X_n = k) \longrightarrow \frac{\lambda_t^k}{k!} e^{-\lambda_t}$.

Limits are unique, so $P(N_t = k) = \frac{\lambda_t^k}{k!} e^{-\lambda_t}$.

<u>Theorem 2</u> (uniqueness). Let (M_t) and (N_t) be Poisson processes $[(A')$ and $(B')]$ with rates λ_1 and λ_2 respectively. Assume $\lambda_1 \neq 0$. Let $L_t = M_{\frac{\lambda_2 t}{\lambda_1}}$ for $0 \le t < \infty$. Then (L_t) and (N_t) have identical finite-dimensional joint distributions.

<u>pf</u> $E(L_1) = \lambda_1 \frac{\lambda_2}{\lambda_1} = \lambda_2$.

Hence for $0 \le s < t < \infty$, $L_t - L_s$ and $N_t - N_s$ are both poisson distributed with parameter $\lambda_2 (t-s)$.

Let $0 \le t_1 < t_2 < \cdots < t_n < \infty$. Recall $(A') \Rightarrow (A)$ for a simple counting process. By independence of increments, the "random vectors"

$N_{t_1} - N_0,\ N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$

and

$L_{t_1} - L_0,\ L_{t_2} - L_{t_1}, \dots, L_{t_n} - L_{t_{n-1}}$

) applying the

$L_{t_1} - L_0,\ L_{t_2} - L_{t_1},\ \dots,\ L_{t_n} - L_{t_{n-1}}$

have the same joint distribution.

applying the same function to two things w/ the same distribution yields two things w/ the same distribution.

Now $N_{t_1} = N_{t_1} - N_0,$

$N_{t_2} = (N_{t_2} - N_{t_1}) + (N_{t_1} - N_{t_0}),$

et cetera

and similarly for L.

So the random vectors

$$N_{t_1},\ N_{t_2},\ \dots,\ N_{t_n}$$

and

$$L_{t_1},\ L_{t_2},\ \dots,\ L_{t_n}$$

have the same joint distribution. $\square$

<u>Fact</u> $\text{law}(X) = \text{law}(Y) \Longrightarrow \text{law}(f(X)) = \text{law}(f(Y))$.

Pf one line.