3/27

Wednesday, March 27, 2019 14:26

<u>Theorem 1</u> Let (N_t) be a Poisson process.

Then there is a number $\lambda \in [0,\infty)$ such that for $0 \le s < t < \infty$, $N_t - N_s$ is Poisson distributed with parameter $\lambda\cdot(t-s)$.

<u>Pf</u> Instead of (A) and (B), we'll use the following weaker assumptions:

(A') for all n, for all $t_0, t_1, \ldots, t_n$,
if $0 < t_0 < t_1 < \cdots < t_n < \infty$,
then the events $\{N_{t_1} - N_{t_0} \geq 1\}$, $\{N_{t_1} - N_{t_2} \geq 1\}, \ldots, \{N_{t_n} - N_{t_{n-1}} \geq 1\}$
are independent.

(B') for all $s, t, u, v \in [0,\infty)$, if $t - s = v - u$,

then $P(N_t - N_s \geq 1) = P(N_v - N_u \geq 1)$.

First, let $t \in [0,\infty)$ and let's show that for some $\lambda_t \in [0,\infty)$, N_t is Poisson distributed w/ parameter λ_t.

For each integer $n \geq 1$, for each $i \in \{1,\ldots,n\}$,

let $\Delta_{ni} = N_{\frac{i}{n}t} - N_{\frac{i-1}{n}t}$.

$A_{ni} = \{\Delta_{ni} \geq 1\}$ = event that (N_u) jumps at least once in the interval $[\frac{i-1}{n}t, \frac{i}{n}t)$.

Let $X_n = \sum_{i=1}^{n} 1_{A_{ni}}$. Clearly $X_n \le N_t$.

since (N_u) is a simple counting process.

Let $\omega \in \Omega$. By assumption, there is a set $J(\omega) \subseteq (0, \infty)$ such that for each $u \in [0, \infty)$, $N_u(\omega) = |J(\omega) \cap (0, \infty]| < \infty$.

In particular, $J(\omega) \cap (0, t]$ is a finite set.

Choose an integer $n_0 \geq 1$ such that $\frac{t}{n_0} < \min\{|u-v| : u, v \in J(\omega) \cap (0, t], u \neq v\}$.

(if $|J(\omega) \cap (0, t]| < 2$, we can just let $n_0 = 1$)

Then for each $n \geq n_0$, $X_n(\omega) = N_t(\omega)$.

(Note: n_0 depends on ω).

Thus $P(X_n \neq N_t) \longrightarrow 0$ as $n \longrightarrow \infty$.

(Remark: By a similar argument, for any simple counting process, $(A') \Longrightarrow (A)$.
To see this, approximate each increment by RVs defined analogously to the X_n's)

Now by (B'), for each n there exists $p_n \in [0, 1]$ such that for each $i \in \{1, \dots, n\}$, $P(A_{ni}) = p_n$. By (A'), for each n, the events $A_{n1}, \dots, A_{nn}$ are independent. Thus X_n has a Binomial distribution with parameter p_n:

$$P(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k} \quad \text{for each } k \in \{0, \dots, n\}.$$

Let $\lambda_t = -\log(P(N_t = 0))$. Then $\lambda_t \in [0, \infty]$.

<u>Claim</u>: $\lambda_t < \infty$ and $np_n \longrightarrow \lambda_t$ as $n \longrightarrow \infty$.

$$P(N_t = 0) = P\left(\bigcap_{i=1}^{n} A_{ni}^c\right) = (1-p_n)^n.$$

Hence $P(N_t = 0) \neq 0$, for otherwise, for each n, $p_n = 1$,

so $P(A_{ni}) = 1$, so $P(X_n = n) = 1$, $P(N_t \geq n) = 1$.

Hence $P(N_t = \infty) = 1$, which is impossible

because $\{N_t < \infty\} = \Omega$.

Since $P(N_t = 0) > 0$, $\lambda_t < \infty$.

$\forall n$

Next, $-n\log(1-p_n) = -\log\left[(1-p_n)^n\right] = -\log(P(N_t = 0)) = \lambda_t < \infty$.

Hence $\log(1-p_n) \longrightarrow 0$ so $p_n \longrightarrow 0$ as $n \longrightarrow \infty$.

But $\lambda_t = -n\log(1-p_n) = -n\left(-p_n - \frac{p_n^2}{2} - \frac{p_n^3}{3} - \cdots\right)$

$$= np_n \underbrace{\left(1 + \frac{p_n}{2} + \frac{p_n^2}{3} + \cdots\right)}_{\downarrow \; 1 \text{ as } n \longrightarrow \infty}$$

So $np_n \longrightarrow \lambda_t$ as $n \longrightarrow \infty$.

This proves the claim

Let $k \in \{0, 1, 2, 3, \ldots\}$.

$P(N_t = k) = P(N_t = k, X_t = N_t) + P(N_t = k, X_t \neq N_t)$.

$P(N_t = k) \geq P(N_t = k, X_t = N_t) \ldots$ (simple way next time)

Fancy way: $P(X_n \neq N_t) \longrightarrow 0$, so $X_n \xrightarrow{P} N_t$, So $\text{law}(X_n) \xrightarrow{w} \text{law}(N_t)$.

But since $np_n \longrightarrow \lambda_t$, $\text{law}(X_n) \xrightarrow{w} \pi(\lambda_t)$.

So $\text{law}(N_t) = \pi(\lambda_t)$.

Now $\lambda_{s+t} = -\log(N_{s+t} = 0) = -\log P[\overbrace{(N_{t+s} - N_t)}^{\geq 0} + \overbrace{N_t}^{\geq 0} = 0]$

$$= -\log P(\underbrace{N_{t+s} - N_t = 0}, \underbrace{N_t = 0})$$

independent events $\longrightarrow$ (Note $\{N_t = 0\} = \{N_t \geq 1\}^c$).

$$= -\log P(N_{t+s} - N_t = 0) - \log P(N_t = 0)$$

$$= -\log P(N_s = 0) - \log P(N_t = 0)$$

$$= \lambda_s + \lambda_t$$

Also, $t \longmapsto \lambda_t = -\log P(N_t = 0)$ is increasing on $[0, \infty)$.

So $\forall t \in [0, \infty)$, $\lambda_t = \lambda \cdot t$ where $\lambda = \lambda_1$.

Finally, for all $s \in (0, \infty)$, the process $(N_{u+s} - N_s)_{0 \leq u < \infty}$ satisfies (A') and (B') and we have

$\lambda_1 = -\log P(N_{1+s} - N_s = 0)$. So $\forall t \geq s$,

$$\text{Law}(N_{t+s} - N_s) = \pi(\lambda_1 t) = \pi(\lambda t).$$

(Remark: In particular, for any simple counting process