3/25

Monday, March 25, 2019 14:18

eg Toss a fair coin 100 times
Estimate the probability that the number of heads obtained is not strictly between 40 and 60

Solution (3 ways)

$X_1, X_2, \dots X_{100}$ indp

$$P(X_i = 1) = \frac{1}{2} = P(X_i = 0)$$

heads tails

$$S_n = X_1 + \dots + X_n$$

$$E(S_n) = \frac{n}{2}. \quad Var(S_n) = \frac{n}{4}, \quad \sigma_{S_n} = \frac{\sqrt{n}}{2}.$$

Specialize to $n = 100$.

$$P(|S_{100} - 50| \geq 10) = P((S_{100} - 50)^2 \geq 100)$$

$$\leq \frac{1}{100} E((S_{100} - 50)^2)$$

markov's inequality

$$= \frac{1}{100} Var(S_{100})$$

$$= \frac{1}{100} \cdot \frac{100}{4} = \frac{1}{4}.$$

We used Chebyshev's Inequality:

$$P(|Z - E(Z)| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} Var(Z).$$

(special case of Markov's inequality).

What if we try Bernstein's Inequality?

$$P(|S_{100} - 50| \geq 10) = P\left(\left|\frac{1}{100} S_{100} - \frac{1}{2}\right| \geq \frac{1}{10}\right)$$

$$\leq 2e^{-n\delta^2/4} \qquad \begin{array}{l} n = 100 \\ \delta = 10 \end{array}$$

$$= 2e^{-1/4}$$

$$= 1.557\ldots > 1$$

(useless!!)

Let's try the Central Limit Theorem

$$P(|S_{100} - 50| \geq 10) = P(|S^*_{100}| \geq 2)$$

$$\approx 1 - \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= 2\int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$\approx 2(0.0228) = 0.0456$

The Poisson Distribution

$\lambda \in [0, \infty)$ $\qquad$ (λ = "intensity")

$$\pi_k(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \text{for } k = 0, 1, 2, \dots$$

$\pi(\lambda)$ is a prob. dist on the non-negative integers because $\pi_k(\lambda) \geq 0$ and

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1.$$

(suppose $P(X=k) = \pi_k(\lambda)$ $\forall k \in \mathbb{N} \cup \{0\}$) $\pi(\lambda)$ has mean λ and variance λ because

$$\hookrightarrow E(X) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$$

And
$$\sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda \left(\frac{d}{dx} x e^x \right) \Big|_{x=\lambda}$$

$$= e^{-\lambda} \lambda \left(e^{\lambda} + \lambda e^{\lambda} \right) = \lambda + \lambda^2$$

So $Var(X) = E(X^2) - E(X)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$.

Also, $$E(X(X-1)) = \sum_{k=0}^{\infty} k(k-1)\, \pi_k(\lambda)$$
$$= \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!}$$
$$= \lambda^2 e^{-\lambda} e^{\lambda}$$
$$= \lambda^2$$

So $E(X^2) - E(X) = \lambda^2$

So $E(X^2) = \lambda^2 + \lambda$

So $Var(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Recall $B_k(n,p) = \binom{n}{k} p^k q^{n-k}$ for $k = 0, 1, 2, \dots, n$,

where $n \in \{0, 1, 2, \dots\}$, $p \in [0,1]$, and $q = 1-p$.

__Theorem__ Let (p_n) be a sequence in $[0,1]$ such that $np_n \longrightarrow \lambda$, where $\lambda \in [0, \infty)$.

Then for each k, $B_k(n, p_n) \xrightarrow[n \to \infty]{\text{as}} \pi_k(\lambda)$

pf $B_k(n, p_n) = \binom{n}{k} p_n^k \, q_n^{n-k}$

$$= \frac{n!}{k!\,(n-k)!} \, p_n^k \, (1-p_n)^{n-k}$$

$$= \frac{1}{k!} \, n(n-1) \cdots (n-k+1) \, p_n^k \, (1-p_n)^{n-k}$$

$$= \frac{(n p_n)^k}{k!} \, \frac{\frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)}{(1-p_n)^k} \, (1-p_n)^n .$$

Now $(1-p_n)^n = \left(1 - \frac{np_n}{n}\right)^n = e^{n \log(1-p_n)}$

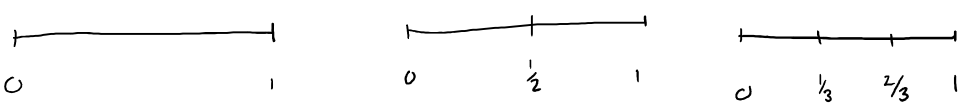
$$= \exp\left(n\left(-p_n + \frac{p_n^2}{2} - \frac{p_n^3}{3} + - \cdots\right)\right)$$

$$= \exp\Big(\underbrace{-np_n}_{\downarrow \, -\lambda} \, \underbrace{\left(1 - \frac{p_n}{2} + \frac{p_n^2}{3} - + \cdots\right)}_{\downarrow \, 1}\Big)$$

$$\longrightarrow e^{-\lambda}.$$

Thus $B_k(n, p_n) \longrightarrow \frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} = \pi_k(\lambda)$.

eg Cosmic Rays

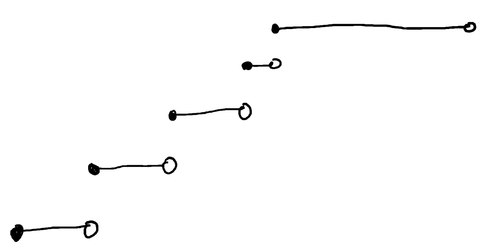


p_n = probability that a cosmic ray arrives in an interval $[\frac{k}{n}, \frac{k+1}{n})$.

# of intervals (of length $\frac{1}{n}$) in which a cosmic ray arrives $\sim B(n, p_n)$.

The Poisson Process

Defn A simple counting process is a family $(N_t)_{0 \le t < \infty}$ of RVs such that for each $\omega \in \Omega$, There is a set $J(\omega) \subseteq (0, \infty)$ such that $\forall\, t \in [0, \infty)$, $N_t(\omega) = |J(\omega) \cap [0, t]| < \infty$.



The paths of a simple counting process are increasing and right cts, with jumps of size 1, and they only change by jumping.

Defn A poisson process is a simple counting process $(N_t)_{0 \le t < \infty}$ such that

(A) (N_t) has independent increments;

i.e. for each n, for all $0 \le t_0 < t_1 < \cdots < t_n < \infty$,

$$N_{t_1} - N_{t_0},\ N_{t_2} - N_{t_1},\ \ldots,\ N_{t_n} - N_{t_{n-1}}$$

are independent.

(B) (N_t) has stationary increments;

i.e. for all $s, t, u, v \in [0, \infty)$, if $t - s = v - u$

Then $\text{law}(N_t - N_s) = \text{law}(N_v - N_u)$.

Remark let X be an RV in $(\mathbb{X}, \mathcal{A})$.

Define μ on $\mathcal{A}$ by $\mu(A) = P(X \in A) = P(X^{-1}[A])$.

Then $\text{law}(X) := \mu$.

a.k.a. the distribution of X or the law of X.

a probability measure on $\mathcal{A}$.

let $f : \mathbb{X} \longrightarrow [0, \infty]$ be mble.

Then $E[f(X)] = \int_{\mathbb{X}} f \, d\,\mathrm{law}(X)$

Let's check this.

① If $f = 1_A$, where $A \in \mathcal{A}$, then $f(X)(\omega) = f(X(\omega))$

$$= \begin{cases} 1 & \text{if } X(\omega) \in A \\ 0 & \text{if } X(\omega) \notin A \end{cases}$$

$$= 1_{X^{-1}[A]}(\omega).$$

So $E[f(X)] = E[1_{X^{-1}[A]}] = P(X^{-1}[A]) = \mathrm{law}(X)(A) = \int_{\mathbb{X}} 1_A \, d\,\mathrm{law}(X)$.

② If f is simple, use linearity of the integral...

③ In the general case, let (f_n) be an increasing sequence of nonnegative simple functions on $\mathbb{X}$ s.t. $f_n \to f$ pointwise. Use MCT.