Let A and B be events with $P(B) \neq 0$.

Then $$P(A \mid B) = \frac{P(AB)}{P(B)}.$$

<u>Remark</u>: A & B are independent if $P(A \mid B) = P(A)$.

<u>Propn 1</u> Let $A_1, \dots, A_n$ be events with $P(A_1 \cdots A_n) \neq 0$.

Then $P(A_1 \cdots A_n) = P(A_1) P(A_2 \mid A_1) P(A_3 \mid A_2 A_1) \cdots P(A_n \mid A_1 \cdots A_{n-1})$.

~~pf~~ "telescoping product."

<u>Propn 2</u> Suppose $A_1, \dots, A_n$ are disjoint events,

$\Omega = \bigcup_{k=1}^{n} A_k$, and $P(A_k) \neq 0$ for $k = 1, \dots, n$.

Then each event B,

$$\underset{\overset{\uparrow}{\dot\bigcup_k BA_k}}{P(B)} = \sum_k \underbrace{P(B \mid A_k) \cdot P(A_k)}_{\downarrow \; P(BA_k)}$$

~~pf~~

<u>Propn 3</u>: Bayes' Theorem (1763). Let $A_1, \dots, A_n$ be disjoint events with $\Omega = \bigcup_k A_k$ and $P(A_k) \neq 0$ $\forall k$.

Let B be an event with $P(B) \neq 0$. Then for each j, $$P(A_j \mid B) = \frac{P(A_j) P(B \mid A_j)}{\sum_k P(A_k) P(B \mid A_k)}.$$

Pf $\sum_k P(A_k) P(B|A_k) = P(B)$.

$$P(BA_j) = P(B) P(A_j | B)$$
$$\parallel \qquad\qquad \parallel$$
$$P(A_j) P(B | A_j) \qquad \square$$

Eg (False positives).

Suppose a lab test on a blood sample yields two results, positive & negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without the disease produce a positive result too. Suppose 1% of people actually have the disease. What is the probability that a person chosen at random from a population will have the disease, given that the person's blood tests positive?

Solution Let D be the event that the person has the disease. We seek $P(D|+)$. $P(D) = 0.01$, $P(+|D) = 0.95$, and $P(+|D^c) = 0.02$. $P(D^c) = 0.99$

$$\text{So } P(D|+) = \frac{P(D) P(+|D)}{P(+|D) P(D) + P(+|D^c) P(D^c)}$$

$$= \frac{0.01 \cdot 0.95}{0.01 \cdot 0.95 + 0.02 \cdot 0.99}$$

$$= \frac{95}{293}$$

$\approx 32\%$

Kolmogorov's Strong Law of Large numbers (~1930)

Let $X_1, X_2, X_3, \ldots$ be iid (independent & identically distributed)

in L^1

integrable real RVs. Then $\dfrac{X_1 + \cdots + X_n}{n} \longrightarrow E(X_1)$ almost surely as $n \to \infty$

Etemadi (~1980) showed that pairwise independence is enough.

The requirement of being identically distributed can also be weakened.

The iid Central Limit Theorem

Let $X_1, X_2, X_3, \ldots$ be iid square-integrable real RVs.

Let $\bar{x} = E(X_1)$, $\sigma = \sqrt{Var(X_1)}$. Let $S_n = X_1 + \cdots + X_n$.

$E(S_n) = n\bar{x}$ and $Var(S_n) = n\sigma^2$

Let $S_n^* = \dfrac{S_n - n\bar{x}}{\sigma\sqrt{n}}$. Then for all real numbers a and b with $a < b$, we have

$$\lim_{n \to \infty} P(a < S_n^* \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

$$\text{law}(S_n^*) \xrightarrow{w} N(0,1).$$

$$E(f(S_n^*)) \longrightarrow \int f(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \quad \text{for each}$$

$$E(f(S_n^*)) \longrightarrow \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \text{for each}$$

bounded continuous function $\mathbb{R} \to \mathbb{R}$.

Levy's Central Limit Theorem

For $n = 1, 2, 3, \ldots$ let $X_1^n, \ldots, X_{k_n}^n$ be independent real RVs.

Suppose for each $\varepsilon > 0$, $\max_j P(|X_j^n| \geq \varepsilon) \longrightarrow 0$.

Let $S_n = \sum_j X_j^n$. Suppose $\text{Law}(S_n) \xrightarrow{w} \mu$

for some Borel probability measure μ on $\mathbb{R}$.

Then $\mu = N(\bar{x}, \sigma^2)$ for some $\bar{x} \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{\geq 0}$

iff $\forall \varepsilon > 0$, $P\left(\max_j |X_j^n| \geq \varepsilon\right) \longrightarrow 0$ as $n \to \infty$.