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<u>Propn</u> Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let $f : X \longrightarrow [0, \infty]$ be mble.

Then $\displaystyle\int f d\mu = \int_0^\infty \mu(f > y)\, dy$.

<u>One "proof"</u>: $\displaystyle\int_X f\, d\mu = \int_X \int_0^\infty 1_{[0, f(x))}(y)\ dy\ d\mu(x)$

$$= \int_0^\infty \int_X 1_{[0, f(x))}(y)\ d\mu(x)\ dy \qquad \text{(provided } \mu \text{ is } \sigma\text{-finite)}$$

$$= \int_0^\infty \int_X 1_{\{f(x) > y\}}\ d\mu(x)\ dy$$

$$= \int_0^\infty \mu(f > y)\ dy.$$

<u>Another proof</u>: ① Suppose f is simple. Let $y_1, \dots, y_n$ be the distinct elements of $f[X]$. Let $A_k = \{f = y_k\}$ for $k = 1, \dots, n$.

Then $X = \dot{\bigcup}_{k=1}^{n} A_k$. So

$$\int_0^\infty \mu(f > y)\ dy = \int_0^\infty \mu\left(\{f > y\} \cap \dot{\bigcup}_{k=1}^{n} A_k\right)\ dy$$

$$= \int_0^\infty \mu\left(\dot{\bigcup}_{k=1}^{n} (\{f > y\} \cap A_k)\right)\ dy$$

$$= \int_0^\infty \sum_{k=1}^{n} \mu\left(\{f > y\} \cap A_k\right)\ dy$$

$$= \sum_{k=1}^{n} \int_0^\infty \mu\left(\{f>y\} \cap A_k\right) dy$$

On A_k, $f = y_k$.

So $\{f>y\} \cap A_k = A_k$ if $y < y_k$

and $\{f>y\} \cap A_k = \emptyset$ if $y \geq y_k$

$$= \sum_{k=1}^{n} \int_0^{y_k} \mu(A_k)\, dy$$

$$= \sum_{k=1}^{n} y_k \mu(A_k)$$

$$= \int f \, d\mu$$

(2) ~~now consider~~ any mble $f : X \longrightarrow [0, \infty]$.

Then there is an increasing sequence (f_n) of simple functions $f_n : X \longrightarrow [0,\infty)$ s.t. $f_n \uparrow f$ pointwise.

Then $\forall$ $y \in [0,\infty)$, $\{f_n > y\} \uparrow \{f > y\}$, so $\mu(f_n > y) \uparrow \mu(f > y)$. Hence

$$\int f \, d\mu \overset{MCT}{=} \lim_{n\to\infty} \int f_n \, d\mu \overset{(1)}{=} \lim_{n\to\infty} \int_0^\infty \mu(f_n > y)\, dy \overset{MCT}{=} \int_0^\infty \mu(f > y)\, dy. \qquad \square$$

<u>Corollary</u> Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $Z : \Omega \longrightarrow [0, \infty]$ be a RV.

Then $E(Z) = \int_0^\infty P(Z > z)\, dz$.

Sums of independent normal RVs

Let X and Y be standard normal RVs. This means that

for each borel set $A \subseteq \mathbb{R}$, $P(X \in A) = \int_A c\, e^{-x^2/2}\, dx$

and $P(Y \in A) = \int_A c\, e^{-y^2/2}\, dy$,

where C is chosen so that $P(X \in \mathbb{R}) = 1$.

(We say X has density φ where $\varphi(x) = c\, e^{-x^2/2}$.
likewise Y has density φ.)

Suppose in addition that X and Y are independent.

Then (X, Y) has density $\Psi(x,y) = \varphi(x)\, \varphi(y) = c^2\, e^{-(x^2+y^2)/2}$ $\qquad (*)$

More detail about $(*)$: $\forall\, A, B \in \text{Borel}(\mathbb{R})$,

$P((X,Y) \in (A,B)) = P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$ because X & Y are indp.

define μ and ν on $\text{Borel}(\mathbb{R}^2)$ by

$\mu(C) = P((X,Y) \in C)$ and $\nu(C) = \iint_C \Psi(x,y)\, dx\, dy$.

Then $\mu(A \times B) = P(X \in A) P(Y \in B) = \left(\int_A \varphi(x)\, dx\right)\left(\int_B \varphi(y)\, dy\right)$

$\iint$ $\quad$ ''

$$\mu(A \times B) = P(X \in A)P(Y \in B) = \left(\int_A \varphi(x)\,dx\right)\left(\int_B \varphi(y)\,dy\right)$$

$$= \int_B \int_A \varphi(x)\varphi(y)\,dx\,dy = \iint_C \psi(x,y)\,dx\,dy = \nu(A \times B).$$

In Particular, $\mu(\mathbb{R}^2) = \nu(\mathbb{R}^2) = 1 < \infty$.

So by the π-λ theorem, $\mu = \nu$.

So $P((X,Y) \in C) = \iint_C \psi(x,y)\,dx\,dy \quad \forall\, C \in \text{Borel}(\mathbb{R}^2)$.

Let's determine c. We know ψ is a probability measure on $\mathbb{R}^2$. Thus

$$1 = \iint_{\mathbb{R}^2} c^2 e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} c^2 e^{-r^2/2}\, r\,dr\,d\theta$$

$$= 2\pi c^2 \int_0^{\infty} e^{-u}\,du = 2\pi c^2$$

So $c = \frac{1}{\sqrt{2\pi}}$ and thus $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Clearly $E(X) = 0 = E(Y)$.

Let's find $\mathrm{var}(X)$ and $\mathrm{var}(Y)$.

Let $\quad R = (X^2+Y^2)^{1/2}. \quad \forall t \geq 0,$

$$P(R^2 > t) = P(R > \sqrt{t})$$

$$= P((X,Y) \text{ lies outside the circle of radius } \sqrt{t} \text{ centered at the origin}).$$

$$= \iint_{|(x,y)|>\sqrt{t}} \psi(x,y)\,dx\,dy$$

$$= \int_0^{2\pi} \int_{\sqrt{t}}^{\infty} \frac{1}{2\pi} e^{-r^2/2}\, r\,dr\,d\theta$$

$$= \int_{t/2}^{\infty} e^{-u}\,du$$

$$= e^{-t/2}.$$

So R^2 has an exponential distribution with parameter $\frac{1}{2}$.

$$E(R^2) = \int_0^{\infty} P(R^2 > t)\,dt = \int_0^{\infty} e^{-t/2}\,dt = 2.$$

So, since $R^2 = X^2 + Y^2$ and X & Y have same distribution,

$$E(X^2) = \frac{1}{2} E(R^2) = 1.$$

So $\operatorname{Var}(X^2) = E(X^2) - (E(X))^2 = 1.$

Similarly, $\operatorname{Var}(Y^2) = 1.$

Now since Ψ, the density of (X, Y), is constant on each circle centered at the origin, and since rotations preserve area, if (X_θ, Y_θ) is the random point obtained by rotating (X, Y) about the origin through an angle θ, then (X_θ, Y_θ) has the same law (prob. distribution) as (X, Y):

$$P((X_\theta, Y_\theta) \in C) = P((X, Y) \in C) \qquad \text{for each} \quad C \in \text{Borel}(\mathbb{R}^2).$$

$(X, Y) = X e_1 + Y e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

A rotation through an angle θ about the origin maps

$$e_1 \quad \text{to} \quad u_1 = (\cos\theta, \sin\theta)$$

$$e_2 \quad \text{to} \quad u_2 = (-\sin\theta, \cos\theta).$$

$(X_\theta, Y_\theta) = X u_1 + Y u_2$.

Hence $X_\theta = X\cos\theta - Y\sin\theta$

Thus for each $\theta \in \mathbb{R}$, $X\cos\theta - Y\sin\theta$ is also standard normal.

Now for any $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$, $\exists\, \theta \in \mathbb{R}$ s.t. $(\alpha, \beta) = (\cos\theta, -\sin\theta)$.

So $\alpha X + \beta Y$ is standard normal.

Now suppose X is normal w/ mean μ and variance a, Now suppose X is normal w/ mean μ and variance a,

and Y is normal with mean ν and variance b.

Let $U = \dfrac{X-\mu}{\sqrt{a}}$. Let $V = \dfrac{Y-\nu}{\sqrt{b}}$.

Suppose X and Y are independent.

Then U and V are independent standard normal RVs.

Let $\alpha = \sqrt{\dfrac{a}{a+b}}$ and $\beta = \sqrt{\dfrac{b}{a+b}}$. $\alpha^2 + \beta^2 = 1$.

So $\alpha U + \beta V$ is standard normal. In other words,

$$\frac{(X-\mu)+(Y-\nu)}{\sqrt{a+b}} = \frac{(X+Y)-(\mu+\nu)}{\sqrt{a+b}}$$ is standard normal.

So $X+Y$ is normal with mean $\mu+\nu$

and variance $a+b$.

<u>Lemma</u> $\log(1+t) = t - \frac{t^2}{2} + R(t)$

where $|R(t)| \le |t|^3$ for $-\frac{2}{3} \le t < \infty$.

pf: $\dfrac{1-u^2}{1+u} = 1-u$ for $u \neq -1$.

Hence $\dfrac{1}{1+u} = 1-u + \dfrac{u^2}{1+u}$ for $u \neq -1$.

hence for $t > -1$, $\log(1+t) = t - \frac{t^2}{2} + R(t)$

hence for $t > -1$, $\log(1+t) = t - \frac{t^2}{2} + R(t)$

where $R(t) = \int_0^t \frac{u^2}{1+u}\, du$.

For $0 \leq t < \infty$, we have $0 \leq R(t) \leq \int_0^t u^2\, du = \frac{t^3}{3}$.

For $-\frac{2}{3} \leq t \leq u \leq 0$, we have $\frac{1}{3} = 1 - \frac{2}{3} \leq 1+u$,

So $0 \leq \frac{u^2}{1+u} \leq 3u^2$, so $0 \leq \int_t^0 \frac{u^2}{1+u}\, du \leq \int_t^0 3u^2\, du = -t^3 = |t|^3$.

So $|R(t)| \leq |t|^3$. $\square$