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<u>Remark</u>: $L^2 \subseteq L^1$

~~pf~~ Let $X \in L^2$. Let $A = \{|X| < 1\}$, $B = \{|X| \geq 1\}$.

Then $|X| = |X| 1_A + |X| 1_B$

$$\leq 1_A + |X|^2 1_B$$

$$\leq 1 + |X|^2$$

depends on the fact that $\int 1 \, dP = 1 < \infty$.

So $E(|X|) = 1 + E(|X|^2) < \infty$,

meaning $X \in L^1$. □

The analogue for an infinite measure space is false.

<u>Remark</u>: Let $X \in L^1$. Then

$$Var(X) = E(X^2) - E(X)^2,$$

because, if we let $\xi = E(X)$, Then

$$E[(X - \xi)^2] = E(\underset{\geq 0}{X^2} - \underset{\text{integrable}}{2\xi X} + \underset{\text{constant}}{\xi^2})$$

$$= E(X^2) - 2\xi E(X) + \xi^2$$

$$= E(X^2) - \xi^2.$$ □

Remark Let $X \in L^1$ and let $a, b \in \mathbb{R}$.

Then $(aX+b)^\circ = aX^\circ$, and so

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).$$

Hence $\sigma_{aX+b} = |a|\,\sigma_X$

Note Let $u, v \in \mathbb{R}$. Since $(u-v)^2 \geq 0$, $2uv \leq u^2+v^2$.

Hence $(u+v)^2 = u^2 + 2uv + v^2 \leq 2(u^2+v^2)$

Hence, for instance, if $X, Y \in L^2$, $X+Y \in L^2$

Remark Let $X \in L^\circ$. For all $a, b \in \mathbb{R}$, we have

$$(X-b)^2 = (X-a+a-b)^2$$

$$\leq 2(X-a)^2 + \underbrace{2(a-b)^2}_{\text{constant}}$$

Thus if $E((X-a)^2) < \infty$, so is $E((X-b)^2)$.

This holds $\forall\, a, b \in \mathbb{R}$. So $\forall\, a \in \mathbb{R}$,

$E((X-a)^2) < \infty$ iff $X \in L^2$. In particular,

$\mathrm{Var}(X) < \infty$ iff $X \in L^2$.

Remark: Let $X \in L^2$. $\forall\, a \in \mathbb{R}$, $E((X-a)^2) \geq \mathrm{Var}(X)$, with equality iff $a = E(X)$.

Pf: As we know, $L^2 \subseteq L^1$ so $X \in L^1$. Let $\xi = E(X)$.

$$E((X-a)^2) = E((X-\xi+\xi-a)^2) = E((X-\xi)^2 + 2(X-\xi)(\xi-a) + (\xi-a)^2)$$

$$= E((X-\xi)^2) + 2(\xi-a)\underbrace{E(X-\xi)}_{0} + (\xi-a)^2$$

Parallel axis Theorem

$$= \mathrm{Var}(X) + (\xi-a)^2. \qquad \square$$

Remark: Let $X, Y \in L^1$. Let $\xi = E(X)$ and $\eta = E(Y)$.

Then $X^\circ Y^\circ = (X-\xi)(Y-\eta) = XY - \eta X - \xi Y + \xi\eta$.

Thus $XY \in L^1$ iff $X^\circ Y^\circ \in L^1$.

Remark if $X, Y \in L^1$ and X and Y are independent,

Then $XY \in L^1$.

Remark if $X, Y \in L^2$, Then $XY \in L^1$ because

$|XY| \le |X|^2 + |Y|^2$ (since $(|X|-|Y|)^2 \ge 0$)

Defn Suppose $X, Y \in L^1$ and $XY \in L^1$. The covariance of X & Y is

$$\mathrm{cov}(X,Y) = E(X^\circ Y^\circ).$$

Remark: Let $X, Y \in L^1$ such that $XY \in L^1$. Then

(a) $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$.

(b) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{cov}(X,Y) + \mathrm{Var}(Y)$.

(c) for all $a, b, c, d \in \mathbb{R}$, $\operatorname{cov}(aX+b, cY+d) = ac \operatorname{Cov}(X,Y)$.

Remark: Let $X, Y \in L^2$. We seek $a, b \in \mathbb{R}$ which minimize

$$E\left[(Y-(aX+b))^2\right].$$

Let's assume that $\operatorname{Var}(X) \neq 0$ (we already handled that case).

Recall that for each $Z \in L^2$, $\operatorname{var}(Z) = E(Z^2) - E(Z)^2$

So $E(Z^2) = \operatorname{Var}(Z) + E(Z)^2$.

Hence for all $a, b \in \mathbb{R}$,

$$\begin{aligned}
E\left[(Y-(aX+b))^2\right] &= E\left[((Y-aX)-b)^2\right] \\
&= \operatorname{Var}(Y-aX) + (E((Y-aX)-b))^2 \\
&= \operatorname{Var}(Y) - 2a\operatorname{cov}(X,Y) + a^2 \operatorname{Var}(X) \\
&\quad + (E((Y-aX)-b))^2 \\
&= \left[a\sigma_X - \frac{\operatorname{cov}(X,Y)}{\sigma_X}\right]^2 + \left[\operatorname{Var}(Y) - \frac{\operatorname{Cov}(X,Y)^2}{\operatorname{Var}(X)}\right] \\
&\quad + (E((Y-aX)-b))^2
\end{aligned}$$

Thus $E(Y-(aX+b))^2$ is minimized exactly when

$$a = \frac{\operatorname{Cov}(X,Y)}{\operatorname{Var}(X)} \quad \text{and} \quad b = E(Y-aX). \tag{1}$$

And its minimum value is $\operatorname{Var}(Y) - \dfrac{\operatorname{Cov}(X,Y)^2}{\operatorname{Var}(X)}$.

Since $E[(Y-(aX+b))^2] \geq 0$, it follows that

$$|\mathrm{cov}(X,Y)| \leq \sigma_X \sigma_Y . \tag{2}$$

equality holds in (2) iff $Y = aX + b$ a.s. for some $a, b \in \mathbb{R}$.

<u>Defn</u> Let $X \in L^2$ with $\mathrm{Var}(X) \neq 0$.

Then $X^* = \dfrac{X - E(X)}{\sigma_X}$ is called "X standardized".

<u>Note</u>: $E(X^*) = 0$, $\sigma_{X^*} = 1$.

$$X = \sigma_X X^* + E(X).$$

<u>Defn</u> Let $X, Y \in L^2$ with $\mathrm{Var}(X) \neq 0$ and $\mathrm{Var}(Y) \neq 0$.

The correlation coefficient of X and Y is

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

Note: $\rho(X,Y) = \mathrm{Cov}(X^*, Y^*)$ so

$$-1 \leq \rho(X,Y) \leq 1.$$

$\rho(X,Y) = 1$ iff $Y = aX + b$ a.s. for some $a > 0$, $b \in \mathbb{R}$.

$\rho(X,Y) = -1$ iff " "$a < 0$" ".

iff $\rho(X,-Y) = 1$.

Remark: Let $X, Y \in L^2$ with $\mathrm{var}(X) \neq 0$ and $\mathrm{var}(Y) \neq 0$.

Then $E\left[(Y^* - (aX^* + b))^2\right]$ is minimized when

$a = \rho(X,Y)$ and $b = 0$.

Defn Let $X, Y \in L^1$. To say X and Y are uncorrelated means $XY \in L^1$ and $E(XY) = E(X)E(Y)$.

Remark: Let $X, Y \in L^1$. Then X & Y are uncorrelated iff $XY \in L^1$ and $\mathrm{Cov}(X,Y) = 0$.

Remark: Let $X, Y \in L^2$. Then X & Y are uncorrelated iff X° and Y° are orthogonal as elements of the inner product space L^2.

Remark: Let $X, Y \in L^1$. Suppose X & Y are independent. Then X and Y are uncorrelated.