$(X, \mathcal{A}, \mu)$

<u>Examples</u>: Let $f : X \longrightarrow [0,\infty]$ be σ-simple. This means $f[X]$ is countable & $\forall y$, $f^{-1}[\{y\}]$ is mble. Note: a σ-simple fn is mble.

Then $$\int f \, d\mu = \sum_y y \cdot \mu(f=y)$$

Pf if f is simple, this was our defn of the integral.

If f is not simple but σ-simple then $f[X]$ is countably infinite.

Let $y_1, y_2, y_3, \ldots$ be the distinct elements of $f[X]$.

Let $\psi_i = y_i \cdot 1_{\{f=y_i\}}$ and $\varphi_n = \sum_{i=1}^n \psi_i$. Then

$\varphi_n \uparrow f$ so $\int f \, d\mu = \lim_{n\to\infty} \sum_{i=1}^n y_i \cdot \mu(f=y_i) = \sum_y y \cdot \mu(f=y)$.

<u>Beppo Levi's Theorem</u>

Let $f_1, f_2, f_3, \ldots \in \mathcal{A}^+$. Let $f = \sum_{k=1}^\infty f_k$. Then $\int f \, d\mu = \sum_{k=1}^\infty \int f_k \, d\mu$

Pf Let $g_n = \sum_{k=1}^n f_k$. Then $g_n \uparrow f$ so $\int f \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \sum_{k=1}^n \int f_k \, d\mu = \sum_{k=1}^\infty \int f_k \, d\mu$.

Integration of $\overline{\mathbb{R}}$-valued fns

Let $f : X \longrightarrow \overline{\mathbb{R}}$.

Define $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$ $\forall x \in X$.

$$f^+ - f^- = f, \qquad f^+ + f^- = |f|.$$

Remark: Let $f : X \longrightarrow \overline{\mathbb{R}}$. f is mble iff f^+ & f^- are mble.

Pf: ($\Rightarrow$) recall that $\max_n \{f_n : n \geq 1\}$ is mble if each f_n is.

($\Leftarrow$) Suppose f^+ & f^- are mble.

Let $0 \leq \varphi_n \uparrow f^+$ and $0 \leq \psi_n \uparrow f^-$ for $\varphi_n, \psi_n \in SF^+$.

Then $\underbrace{\varphi_n - \psi_n}_{\substack{\text{simple,} \\ \mathbb{R}\text{-valued}}} \longrightarrow f$ so f is a limit of mble fns.

Defn: Let $f : X \longrightarrow \overline{\mathbb{R}}$.

(a) to say $\int f d\mu$ is defined means f is mble and $\int f^+ d\mu$ and $\int f^- d\mu$ are not both ∞.

(b) if $\int f d\mu$ is defined then $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$.

(c) to say f is μ-$\int$ble means $\int f d\mu$ is defined & finite.

Warning: $\int_0^\infty \left(\frac{\sin x}{x}\right)^+ dx = \infty$, and so is $\int_0^\infty \left(\frac{\sin x}{x}\right)^- dx$.

So $\int_0^\infty \frac{\sin x}{x}\, dx$ is not defined.

What is defined is $\int_0^{\to\infty} \frac{\sin x}{x}\, dx = \lim_{b \to \infty} \int_0^b \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

Remark Let $f : X \longrightarrow \overline{\mathbb{R}}$. Suppose $\int f \, d\mu$ is defined.

Then $\left|\int f \, d\mu\right| \le \int |f| \, d\mu$

since $\left|\int f \, d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right| \le \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right| = \int f^+ d\mu + \int f^- d\mu = \int f^+ + f^- \, d\mu = \int |f| \, d\mu$.

Remark: Let $f : X \longrightarrow \overline{\mathbb{R}}$. Then f is μ-$\int$ble iff f is mble & $\int |f| \, d\mu < \infty$.

Propn Let $f : X \longrightarrow \overline{\mathbb{R}}$. Suppose $\int f \, d\mu$ is defined. Let $c \in \mathbb{R}$.

then $\int cf \, d\mu$ is defined & $\int cf \, d\mu = c \int f \, d\mu$.

pf $cf = cf^+ - cf^-$

Warning not if $c = \pm\infty$.

Theorem: $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$ when $f, g : X \longrightarrow (-\infty, \infty]$ are mble

and $\int f^- d\mu$ and $\int g^- d\mu$ are both finite.

Lemma: Suppose $h = h_1 - h_2$ where $h_1 : X \longrightarrow [0, \infty]$ and $h_2 : X \to [0, \infty)$ are mble

and $\int h_1 \, d\mu - \int h_2 \, d\mu$ is defined.

Then $\int h_1 \, d\mu - \int h_2 \, d\mu = \int h \, d\mu$

pf $h^+(x) = \max\{h(x), 0\} = \max\{h_1(x) - h_2(x), 0\} \le \max\{h_1(x), 0\} = h_1(x)$,

Similarly $h^-(x) \le h_2(x)$. Thus $\int h^- d\mu \le \int h_2 \, d\mu < \infty$.

now $h_1 - h_2 = h = h^+ - h^-$ so $h_1 + h^- = h^+ + h_2$ so

$\int h_1 \, d\mu + \int h^- d\mu = \int h^+ d\mu + \int h_2 \, d\mu$ so $\int h_1 \, d\mu - \int h_2 \, d\mu = \int h^+ d\mu - \int h^- d\mu = \int h \, d\mu$.

Integration of $\mathbb{R}^d$-valued fns

Let $f: X \longrightarrow \mathbb{R}^d$. Then $f(x) = (f_1(x), \dots, f_d(x))$ where each $f_i : X \to \mathbb{R}$.

Lemma: f is mble iff $f_1, \dots, f_d$ are all mble.

Pf ($\Rightarrow$) Suppose f is mble.

$$\{f_k > y\} = f^{-1}[\mathbb{R} \times \cdots \times \mathbb{R} \times (y, \infty) \times \mathbb{R} \times \cdots \times \mathbb{R}] \in \mathcal{A}.$$

($\Leftarrow$) suppose $f_1, \dots, f_d$ are mble.

$$f^{-1}[\underbrace{(a_1, b_1) \times \cdots \times (a_d, b_d)}_{(a,b)}] = \bigcap_{k=1}^{d} f_k^{-1}[(a_k, b_k)] \in \mathcal{A}.$$

Let G be an open $\subseteq \mathbb{R}^d$. Let $\mathcal{U} = \{(a,b) \subseteq G : a = (a_1, \dots, a_d) \in \mathbb{Q}^d, b \in \mathbb{Q}^d \text{ too}\}$

Then $\cup \mathcal{U} = G$. Hence $f^{-1}[G] = \cup\{f^{-1}[I] : I \in \mathcal{U}\} \in \mathcal{A}$.