Lemma Let $\varphi \in SF^+$. Define ν on $\mathcal{A}$ by $\nu(A) = \int_A \varphi \, d\mu$.

Then ν is a measure on $\mathcal{A}$.

Pf Let the distinct elements of $\varphi[X]$ be $y_1, \dots, y_n$.

Let $B_k = \{\varphi = y_k\}$. Then $\dot\bigcup B_k = X$, and $\varphi = \sum y_k 1_{B_k}$

$$\nu(A) = \int_A \varphi \, d\mu = \sum y_k \int 1_{A \cap B_k} \, d\mu = \sum y_k \, \mu(A \cap B_k).$$

for each k, the fn $A \mapsto \mu(A \cap B_k)$ is a measure on $\mathcal{A}$.

$$\left(\left(\dot\bigcup_{j=1}^{\infty} A_j \right) \cap B_k = \dot\bigcup_{j=1}^{\infty} (A_j \cap B_k) \quad \text{etc} \right).$$

So $A \longmapsto y_k \, \mu(A \cap B_k)$ is too,

So $A \mapsto \sum_{k=1}^{\infty} y_k \, \mu(A \cap B_k)$ is too

The Monotone Convergence Theorem

Let $f_1 \le f_2 \le f_3 \le \cdots$ be $[0, \infty]$-valued mble fns on X.

Let $f = \lim_{n \to \infty} f_n$. Then $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$.

KNOW THIS:

Pf: $\int f_1 \, d\mu \le \int f_2 \, d\mu \le \int f_3 \, d\mu \le \cdots$. Let $L = \lim \int f_n \, d\mu$.

each $f_n \le f$ so $L \le \int f \, d\mu$. Let's show $\int f \, d\mu \le L$.

Let $t < \int f \, d\mu$. we'll show that $t < \int f_n \, d\mu$ for some n.

by defn, $\int f\,d\mu = \sup\{\int \varphi\,d\mu : \varphi \in SF^+,\ \varphi \le f\}$

Hence for some $\varphi \in SF^+$ w/ $\varphi \le f$, we have $t < \int \varphi\,d\mu$.

Let $\alpha \in (0,1)$ s.t. $t < \alpha \int \varphi\,d\mu$. Let $\psi = \alpha \cdot \varphi$.

Then $\psi \in SF^+$ and $t < \int \psi\,d\mu$. And $\forall x \in X$, if $0 < f(x) < \infty$

then $\psi(x) < f(x)$. While if $f(x) = \infty$ then $\psi(x) < f(x)$ too.

So $\psi < f$ on $\{f > 0\}$. Let $A_n = \{f_n \geq \psi\}$.

Then $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$.

Claim: $\bigcup_{n=1}^{\infty} A_n = X$.

If $f(x) > 0$ then $\psi(x) < f(x)$. Since $f_n(x) \uparrow f(x)$, $\exists n$ s.t. $\psi(x) < f_n(x)$. So $x \in A_n$.

If $f(x) = 0$ then each $f_n(x) = 0$ and $\psi(x) = 0$ so $x \in A_n\ \forall n$.

Define ν on $\mathcal{A}$ by $\nu(A) = \int_A \psi\,d\mu$. Then ν is a measure on $\mathcal{A}$.

$$\int f_n\,d\mu \geq \int 1_{A_n} f_n\,d\mu \geq \int 1_{A_n} \psi\,d\mu = \int_{A_n} \psi\,d\mu = \nu(A_n).$$

As $n \to \infty$, $\nu(A_n) \longrightarrow \nu(X) = \int \psi\,d\mu > t$,

So $L = \lim_{n\to\infty} \int f_n\,d\mu > t$.

t was arbitrary so $L \geq \int f\,d\mu$. $\square$

<u>Fatou's Lemma</u>: Let $f_1, f_2, f_3, \ldots \in \mathcal{A}^+$, and let $f = \liminf_{n\to\infty} f_n$.

Then $\int f \, d\mu \le \liminf\limits_{n\to\infty} \int f_n \, d\mu$.

<u>proof</u>: Let $g_n = \inf\limits_{k \ge n} f_k$. Then $g_1 \le g_2 \le g_3 \le \cdots$

and $\lim\limits_{n\to\infty} g_n = f$. So $\lim\limits_{n\to\infty} \int g_n \, d\mu = \int f \, d\mu$ by MCT above.

$\forall n$, $g_n \le f_n$ so $\int g_n \, d\mu \le \int f_n \, d\mu$, so

$$\lim_{n\to\infty} \int g_n \, d\mu = \liminf_{n\to\infty} \int g_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \qquad \square$$

<u>Corollary 1</u>: Let $f, f_1, f_2, f_3, \ldots \in \mathcal{A}^+$; suppose $f_n \to f$ pointwise and $0 \le f_n \le f$. Then $\int f_n \, d\mu \to \int f \, d\mu$.

<u>Pf</u> By Fatou's lemma, $\liminf\limits_{n\to\infty} f_n = f$ so $\int f \, d\mu \le \liminf\limits_{n\to\infty} \int f_n \, d\mu$

but $\int f_n \, d\mu \le \int f \, d\mu$ $\forall n$ so $\int f \, d\mu \ge \limsup\limits_{n\to\infty} \int f_n \, d\mu$.

But $\liminf \le \limsup$ so they must be equal & equal $\int f \, d\mu$.

<u>Corollary 2</u>: Suppose $f, f_1, f_2, f_3, \ldots \in \mathcal{A}^+$.

Suppose $f_n \to f$ pointwise & $\int f_n \, d\mu \to L < \infty$.

then $\int f \, d\mu < \infty$.

<u>Pf</u> $\int f \, d\mu \overset{\text{fatou}}{\le} \liminf\limits_{n\to\infty} \int f_n \, d\mu < \infty$

<u>Corollary 4</u> Let $f, f_1, f_2, f_3, \ldots : X \to \mathbb{C}$ be mble with $f_n \to f$ pointwise.

Let $g \in \mathcal{A}^+$. Suppose $|f_n| < g$ $\forall n$ and $\int g \, d\mu < \infty$.

Then $\int |f - f_n| \, d\mu \longrightarrow 0$

<u>Proof</u>: Note that $|f| = \lim_{n\to\infty} |f_n| \leq g$. Let $h_n = 2g - |f - f_n|$.

$h_n \in \mathcal{A}^+ \ \forall n$ ($|f - f_n| \leq |f| + |f_n| \leq g + g = 2g$, so $h_n \geq 0$).

$h_n \longrightarrow 2g$ and $h_n \leq 2g$ so $\int h_n \, d\mu \longrightarrow \int 2g \, d\mu$

Now $h_n + |f - f_n| = 2g$ so $\int h_n d\mu + \int |f - f_n| d\mu = \int 2g \, d\mu$

So $\int |f - f_n| d\mu = \int 2g \, d\mu - \int h_n \, d\mu \longrightarrow \int 2g \, d\mu - \int 2g \, d\mu = 0$.

<u>Corollary 3</u>: Let $f, g \in \mathcal{A}^+$. Then $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

<u>Proof</u>: Since $f, g \in \mathcal{A}^+$, $\exists$ increasing sequences $(\varphi)_n$ and (ψ_n) in SF^+ s.t. $\varphi_n \uparrow f$ and $\psi_n \uparrow g$. Then $\varphi_n + \psi_n \uparrow f + g$.

$$
\begin{aligned}
\text{So } \int f + g \, d\mu &= \lim_{n\to\infty} \int \varphi_n + \psi_n \, d\mu \qquad \text{(MCT)} \\
&= \lim_{n\to\infty} \left(\int \varphi_n d\mu + \int \psi_n d\mu \right) \\
&= \lim_{n\to\infty} \int \varphi_n d\mu + \lim_{n\to\infty} \int \psi_n \, d\mu \\
&= \int f \, d\mu + \int g \, d\mu \qquad \text{(MCT } \times 2\text{)}.
\end{aligned}
$$