$(X, \mathcal{A}, \mu)$ mble space.

<u>Propn</u>: Let $f, g : X \longrightarrow [0, \infty]$ be mble.

Then (a) $f+g$ & $f \cdot g$ are mble,

and $\frac{f}{g}$ is mble where $g \neq 0$.

(b) $\{f<g\}$, $\{f>g\}$, $\{f \leq g\}$, $\{f \geq g\}$, and $\{f=g\}$ are mble

Pf (a) Let φ_n & ψ_n be simple fns s.t. $\varphi_n \uparrow f$ & $\psi_n \uparrow g$.

Then $\varphi_n + \psi_n$ is simple & $\varphi_n + \psi_n \longrightarrow f+g$, et. cetera.

(b) $\{f<g\} = \bigcup_{q \in \mathbb{Q}} (\{f<q\} \cap \{q<g\})$.

$\{f \geq g\} = X \setminus \{f<g\}$.

$\{f=g\} = \{f \geq g\} \cap \{f \leq g\}$.

$\{f \neq g\} = X \setminus \{f=g\}$.

<u>Note</u>: actually, we've just shown that (b) holds for mble fns $X \longrightarrow [-\infty, \infty]$.

A Wild mble fn.

$1_{\mathbb{Q}}$ is not wild enough

for $0<x<1$, let $f(x)$ be the long-run upper frequency of 1's in the standard binary expansion of x.

then $f[(a,b)] = [0,1]$ $\forall$ $0<a<b<1$.

f is Borel measurable bc its a limsup of mble functions. (in fact, simple!)

f is not Riemann $\int$ble.

but it is Lebesgue $\int$ble.

<u>Remark</u>: Let $\varphi, \psi : X \longrightarrow [0,\infty)$ be simple with $\varphi \leq \psi$.

then $\int \varphi \, d\mu \leq \int \psi \, d\mu$.

<u>Pf</u>: $\psi = \varphi + \underbrace{(\psi - \varphi)}_{\text{simple, } \geq 0}$, so $\int \psi \, d\mu = \int \varphi \, d\mu + \overbrace{\int (\psi + \varphi) \, d\mu}^{\geq 0} \geq \int \varphi \, d\mu$.

<u>Notation</u>:

$$SF^+ = \{\varphi : \varphi \text{ is a } [0,\infty)\text{-valued simple fn on } X\}$$
$$\|$$
$$SF^+(X, \mathcal{A})$$

Suppose $f \in SF^+$. Then $\int f \, d\mu = s \{\int \varphi \, d\mu : \varphi \in SF^+, \varphi \leq f\}$

from previous remark.

<u>Defn</u>: Let $f : X \longrightarrow [0,\infty]$ be mble ("let $f \in \mathcal{A}^+$").

Then $\int f \, d\mu = \sup \{\int \varphi \, d\mu : \varphi \in SF^+ \;\&\; \varphi \leq f\}$

Markov's Inequality:

Let $f \in \mathcal{A}^+$ and let $0 < y < \infty$.

then $\mu(f \geq y) \leq \frac{1}{y} \int f \, d\mu$

Pf $\underbrace{y\, 1_{\{f \le y\}}}_{SF^+} \le f$

so $\displaystyle\int y\, 1_{\{f \le y\}}\, d\mu \le \int f\, d\mu$

so $\displaystyle\mu(f \le y) \le \frac{1}{y} \int f\, d\mu.$

<u>Propn</u>: Let $f \in \mathcal{A}^+$. Suppose $\int f\, d\mu < \infty$.

then $\mu(f = \infty) = 0$.

Pf for $0 < y < \infty$,

$\{f = \infty\} \subset \{f \ge y\}$, and $\displaystyle\lim_{y \to \infty} \mu(f \ge y) \le \lim_{y \to \infty} \frac{1}{y} \int f\, d\mu = 0.$

so $\mu(f = \infty) = 0$.

<u>Propn</u>: Let $f \in \mathcal{A}^+$. Then $\int f\, d\mu = 0$ iff $\mu(f > 0) = 0$.

(i.e. "$f = 0$ μ-a.e.").

Pf ($\Longrightarrow$) Suppose $\int f\, d\mu = 0$. then for $0 < y < \infty$,

$\displaystyle 0 \le \mu(f \ge y) \le \frac{1}{y} \int f\, d\mu = 0$

countable sub-additivity

So $\displaystyle\mu(f > 0) = \mu\Big(\bigcup_{n=1}^{\infty} \{f \ge \tfrac{1}{n}\}\Big) \le \sum_{n=1}^{\infty} \mu(f > \tfrac{1}{n}) = 0.$

<u>Propn</u> Let $f \in \mathcal{A}^+$. Suppose $\int f\, d\mu < \infty$. then $\{f > 0\}$

is of σ-finite μ-measure.

<u>Pf</u>: $\displaystyle\{f > 0\} = \bigcup_{n \in \mathbb{N}} \{f \ge \tfrac{1}{n}\}$ and for each $n \in \mathbb{N}$, $\mu(f \ge \tfrac{1}{n}) < \infty$.

Propn: Let $f \in \mathcal{a}^+$ and $c \in [0,\infty]$.

Then $\int cf\,d\mu = c\int f\,d\mu$

Pf either $c=0$, $0<c<\infty$, or $c=\infty$.

Case 0: Suppose $c=0$. Then $cf=0$ so $\int cf\,d\mu = 0$.

and $c\int f\,d\mu = 0$ even if $\int f\,d\mu = \infty$.

Case 1: Suppose $0<c<\infty$. Then $\{\psi \in SF^+ : \psi \le cf\} = \{c\varphi \in SF^+ : \varphi \le f\}$.

$\forall \varphi \in SF^+$, $\int c\varphi\,d\mu = c\int \varphi\,d\mu$.

Taking the sup gives the result:

Let $t < \int cf\,d\mu$. Then for some $\psi \in SF^+$ with $\psi \le cf$,

we have $\int \psi\,d\mu > t$. Let $\varphi = \frac{1}{c}\psi$, so

$\int \varphi\,d\mu \le \int f\,d\mu$, so

$$t < \int \psi\,d\mu = c\int \varphi\,d\mu \le c\int f\,d\mu.$$

This holds $\forall t < \int cf\,d\mu$, so $\int cf\,d\mu \le c\int f\,d\mu$.

Same thing works the other way.

Case 2: Suppose $c=\infty$. Then $cf = \infty 1_A$ where $A = \{f>0\}$.

if $\mu(A)=0$ then $f=0$ a.e. so $cf=0$ a.e.,

So $\int cf\,d\mu = 0 = c\int f\,d\mu$.

If $\mu(A)>0$ then $\int f\,d\mu > 0$ so $c\int f\,d\mu = \infty$.

And $\int cf\,d\mu \ge n\mu(A) \longrightarrow \infty$ as $n\to\infty$, since $n1_A \le \infty 1_A$.

recall if $f=0$ a.e. then $\int f\,d\mu$.

Let $\varphi \in SF^+$ with $\varphi \le f$. for each $y > 0$,

$\{\varphi = y\} \subseteq \{f > 0\}$, so $\mu(\varphi = y) = 0$,

So $\int \varphi \, d\mu = 0$, So $\int f \, d\mu = \sup 0 = 0$.

<u>Propn</u> Let $f \in \mathcal{a}^+$, let $A, B \in \mathcal{a}$ with $A \cap B = \emptyset$.

then $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$

(Recall that $\int_E f \, d\mu = \int f \cdot 1_E \, d\mu$).

<u>Pf</u> $1_{A \cup B} = 1_A + 1_B$. Let $\varphi \in SF^+$ with $\varphi \le 1_{A \cup B} f$.

Let $\varphi_1 = 1_A \varphi$ and $\varphi_2 = 1_B \varphi$. Then $\varphi_1, \varphi_2 \in SF^+$ & $\varphi_1 \le 1_A f$, $\varphi_2 \le 1_B f$,

and $\varphi_1 + \varphi_2 = \varphi$ so

$$\int \varphi \, d\mu = \int \varphi_1 \, d\mu + \int \varphi_2 \, d\mu \le \int 1_A f \, d\mu + \int 1_B f \, d\mu$$

So $\int_{A \cup B} f \, d\mu \le \int_A f \, d\mu + \int_B f \, d\mu$.

Now we prove $\ge$. if $\int_{A \cup B} f \, d\mu = \infty$, we are done.

Suppose $\int_{A \cup B} f \, d\mu < \infty$. Then $\int_A f \, d\mu$, $\int_B f \, d\mu < \infty$.

Let $\varepsilon > 0$. for some $\varphi_1, \varphi_2 \in SF^+$ w/ $\varphi_1 \le 1_A f$, $\varphi_2 \le 1_B f$,

we have $\int \varphi_1 \, d\mu > \int 1_A f \, d\mu - \frac{\varepsilon}{2}$, $\int \varphi_2 \, d\mu > \int 1_B f \, d\mu - \frac{\varepsilon}{2}$.

Let $\varphi = \varphi_1 + \varphi_2$. Then $\varphi \le 1_{A \cup B} f$, and $\int \varphi \, d\mu > \int_A f \, d\mu + \int_B f \, d\mu - \varepsilon$.

So $\int_{A \cup B} f \, d\mu \ge \int_A f \, d\mu + \int_B f \, d\mu$.

Let $f_1, f_2, \ldots \in \mathcal{a}^+$ with $f_n \uparrow f$.

Then $\int f_n \, d\mu \uparrow \int f \, d\mu$.