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Thm (The grouping Thm for RVs) Let $((\mathbb{X}_i, \mathcal{A}_i))_{i\in I}$ be a family of mble spaces. For each $i \in I$, let $X_i : \Omega \longrightarrow \mathbb{X}_i$ be a RV. Suppose the family $(X_i)_{i\in I}$ is independent. Let $(I_j)_{j\in J}$ be a disjoint family of subsets of I. $\forall j \in J$, let $\mathbb{Y}_j = \prod_{i\in I_j} \mathbb{X}_i$, let $\mathcal{B}_j = \bigotimes_{i\in I_j} \mathcal{A}_i$, and define $Y_j : \Omega \longrightarrow \mathbb{Y}_j$ by $Y_j(\omega) = (X_i(\omega))_{i\in I_j}$. Then the family $(Y_j)_{j\in J}$ is independent.

Pf For each $i \in I$, let $\mathcal{H}_i = \sigma(X_i) = X_i^{-1}[\mathcal{A}_i] = \{X_i^{-1}[A] : A \in \mathcal{A}_i\}$.

Then $(\mathcal{H}_i)_{i\in I}$ is an independent family of π-systems (in fact, σ-fields) on Ω.

For each $j \in J$, Let $\mathcal{E}_j = \sigma(H : H \in \mathcal{H}_i \text{ for some } i \in I_j)$.

Then, by the grouping thm for π-systems, the family of σ-fields $(\mathcal{E}_j)_{j\in J}$ is independent. But $\mathcal{E}_j = \sigma(X_i^{-1}[A] : i \in I_j, A \in \mathcal{A}_i)$.

So $\mathcal{E}_j = \sigma\left(\bigcup_{i\in I_j} \sigma(X_i)\right) = \sigma(Y_j)$, and so $(Y_j)_{j\in J}$ is independent.

eg Let $X_1, X_2, X_3, X_4, X_5 : \Omega \longrightarrow \mathbb{R}$ be independent RVs.

Define $Y_1 : \Omega \longrightarrow \mathbb{R}$ and $Y_2 : \Omega \longrightarrow \mathbb{R}^3$ by

$$Y_1(\omega) = (X_1(\omega), X_3(\omega)), \quad Y_2(\omega) = (X_2(\omega), X_4(\omega), X_5(\omega)).$$

Then Y_1 & Y_2 are independent.

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Hence $X_1 + X_3$ and $X_2 - X_4 e^{X_5}$ are independent as well.

$X_1 + X_3$ is $\sigma(Y_1)$-mble.

$X_2 - X_4 e^{X_5}$ is $\sigma(Y_2)$-mble.

Theorem Let $X, Y : \Omega \longrightarrow [0,\infty]$ be independent RVs.
Then $E(XY) = E(X)\cdot E(Y)$.

Pf ① Suppose $X = 1_A$ and $Y = 1_B$, where $A, B \in \mathcal{F}$. Then $XY = 1_{AB}$ and A & B are independent, so $E(XY) = P(AB) = P(A)P(B) = E(X)E(X)$.

② Suppose X and Y are simple. Let $a_1, \dots, a_m$ be the distinct values of X and $b_1, \dots, b_n$ be the distinct values of Y.

Let $A_j = \{X = a_j\} = X^{-1}[\{a_j\}]$ and $B_k = \{Y = b_k\} = Y^{-1}[\{b_k\}]$.

Then $X = \sum_{j=1}^{m} a_j 1_{A_j}$ and $Y = \sum_{k=1}^{n} b_k 1_{B_k}$. For each pair (j,k), the events A_j and B_k are independent because $A_j \in \sigma(X)$, $B_k \in \sigma(Y)$.

So $E(XY) = E\left[\left(\sum_{j=1}^{m} a_j 1_{A_j}\right)\left(\sum_{k=1}^{n} b_k 1_{B_k}\right)\right] = \sum_{j=1}^{m}\sum_{k=1}^{n} a_j b_k E(1_{A_j} 1_{B_k})$

$$= \sum_{j=1}^{m}\sum_{k=1}^{n} a_j b_k E(1_{A_j}) E(1_{B_k}) = \left(\sum_{j=1}^{m} a_j E(1_{A_j})\right)\left(\sum_{k=1}^{n} b_k E(1_{B_k})\right)$$

$$= E(X)E(Y).$$

③ The general case. When X is $\sigma(X)$-mble, $\exists$ an increasing sequence of $\sigma(X)$-mble fns $X_n : \Omega \longrightarrow [0,\infty)$ s.t. $\forall \omega \in \Omega$, $X_n(\omega) \uparrow X(\omega)$ as $n \to \infty$. Similarly, $\exists$ an increasing sequence of $\sigma(Y)$-mble fns $Y_n : \Omega \longrightarrow [0,\infty)$ s.t. $\forall \omega \in \Omega$, $Y_n(\omega) \uparrow Y(\omega)$ as $n \to \infty$.

For each n, $E(X_n Y_n) = E(X_n) E(Y_n)$

$\quad\quad\quad\quad\quad\quad |\quad\quad\quad\quad\quad \downarrow$ as $n \to \infty$ (apply MCT).

For each n,

$$\begin{array}{cccc} & E(X_nY_n) & = & E(X_n)E(Y_n) \\ & \downarrow & & \downarrow \\ \text{So} & E(XY) & = & E(X)E(Y). \end{array}$$

as $n \to \infty$ (apply MCT).

<u>Note</u>: If $a_n \to 0$ and $b_n \to \infty$, it doesn't mean that $a_n b_n \to 0 = 0 \cdot \infty$.

But this doesn't affect the preceding proof.

<u>Corollary</u>: Let $X, Y, Z : \Omega \to [0, \infty]$ be independent RVs.

Then $E(XYZ) = E(X)E(Y)E(Z)$.

~~pf~~. XY and Z are independent$_{\text{by the grouping theorem}}$ so $E(XYZ) = E(XY)E(Z)$.

Similarly: Let $X_1, \dots, X_n : \Omega \to [0, \infty]$ be indp. Then $E(\prod X_i) = \prod E(X_i)$

<u>Corollary</u>: Let $X, Y : \Omega \to \mathbb{R}$ be independent RVs with $E(|X|), E(|Y|) < \infty$. Then $E(|XY|) < \infty$ and $E(XY) = E(X)E(Y)$.

~~pf~~ $E|XY| = E|X|\,E|Y| < \infty$ by previous corollary.

Now $X = X^+ - X^-$ and $Y = Y^+ - Y^-$ so

$$\begin{aligned} E(XY) &= E(X^+Y^+ - X^+Y^- - X^-Y^+ + X^-Y^-) \\ &= E(X^+Y^+) - E(X^+Y^-) - E(X^-Y^+) + E(X^-Y^-) \\ &= E(X^+)E(Y^-) - E(X^+)E(Y^-) - E(X^-)E(Y^+) + E(X^-)E(Y^-) \\ &= (E(X^+) - E(X^-))(E(Y^+) - E(Y^-)) = E(X)E(Y). \end{aligned}$$

Similarly: Let $X_1, \dots, X_n : \Omega \longrightarrow \mathbb{R}$ be indp. with $E|X_i| < \infty$. Then $E(\prod X_i) = \prod E(X_i)$.

Then $E|\prod X_i| < \infty$.

Notation: L^0 = the set of all Real RVs on Ω.

For $0 < p < \infty$, $L^p = \{ X \in L^0 : \underbrace{E(|X|^p)}_{= \int_\Omega |X|^p dP} < \infty \}$

Thus $L^1 = \{ X \in L^0 : E(|X|) < \infty \}$.

$L^2 = \{ X \in L^0 : E(|X|^2) < \infty \}$.

Defn: Let $X \in L^1$. $X^\circ = X - E(X)$ is "X centered." $E(X^\circ) = 0$.

Note: Let $X, Y \in L^1$. Then $(X+Y)^\circ = X^\circ + Y^\circ$.

Defn Let $X \in L^1$. Then the variance of X is $Var(X) = E[(X^\circ)^2]$.

The standard deviation of X is $\sigma_X = \sqrt{Var(X)} = \|X^\circ\|_2$.