Products of Mble Spaces.

Let $(X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2)$ be mble spaces.

Let $X = X_1 \times X_2$.

Define $\mathcal{A}_1 \otimes \mathcal{A}_2$ to be the σ-field on X generated by $\mathcal{A}_1 \odot \mathcal{A}_2 = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$.

$\pi_1 : X \longrightarrow X_1 \qquad \pi_1(x_1, x_2) = x_1$

$\pi_2 : X \longrightarrow X_2 \qquad \pi_2(x_1, x_2) = x_2$

for all $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, $\pi_1^{-1}[A_1] \cap \pi_2^{-1}[A_2] = A_1 \times A_2$.

Thus $\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma\left(\pi_i^{-1}[A] : i \in \{1,2\}, A \in \mathcal{A}_i\right)$.

<u>Thm</u>: Let $\mathcal{A}_1 = \mathcal{A}_2 = \text{Borel}(\mathbb{R})$.

Then $\mathcal{A}_1 \otimes \mathcal{A}_2 = \text{Borel}(\mathbb{R}^2)$.

<u>pf idea</u>: any open subset of $\mathbb{R}^2$ is a <u>countable</u> union of open rectangles.

Thm: Let X and Y be topological space.

Let $\mathcal{A} = \text{Borel}(X)$, $\mathcal{B} = \text{Borel}(Y)$, $Z = X \times Y$, and $\mathcal{C} = \text{Borel}(Z)$.

Let $\mathcal{R} = \{U \times V : U \text{ open} \subseteq X,\ V \text{ open} \subseteq Y\}$. Then

(a) $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{R}) \subseteq \mathcal{C}$

(b) if each open subset of Z is a union of countably many open rectangles, then $\mathcal{A} \otimes \mathcal{B} = \mathcal{C}$.

(c) in particular, if X or Y is $\underbrace{\text{second countable}}$, then $\mathcal{A} \otimes \mathcal{B} = \mathcal{C}$.

this means that there is some countable collection of open sets such that each open set is a union of the sets it contains from this collection.

Pf (a) $\mathcal{R} \subseteq \mathcal{A} \otimes \mathcal{B}$, so $\sigma(\mathcal{R}) \subseteq \mathcal{A} \otimes \mathcal{B}$. For each U open $\subseteq X$, $\pi_1^{-1}[U] = U \times Y \in \mathcal{R} \subseteq \sigma(\mathcal{R})$. Let $\tilde{\mathcal{A}} = \{A \subseteq X : \pi_1^{-1}[A] \in \sigma(\mathcal{R})\}$. Then each open $\subseteq X$ belongs to $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{A}}$ is a σ-field on X so $\tilde{\mathcal{A}} \supseteq \text{Borel}(X) = \mathcal{A}$. Similarly, $\tilde{\mathcal{B}} = \{B \subseteq Y : \pi_2^{-1}[B] \in \sigma(\mathcal{R})\} \supseteq \mathcal{B}$.

So $\forall\ A \in \mathcal{A}$, $B \in \mathcal{B}$, $\pi_1^{-1}[A] \in \sigma(\mathcal{R})$ and $\pi_2^{-1}[B] \in \sigma(\mathcal{R})$,

so $\pi_1^{-1}[A] \cap \pi_2^{-1}[B] = A \times B \in \sigma(\mathcal{R})$, so $\sigma(\mathcal{R})$ contains $\mathcal{A} \odot \mathcal{B}$,

and so $\sigma(\mathcal{R}) \supseteq \sigma(\mathcal{A} \odot \mathcal{B}) = \mathcal{A} \otimes \mathcal{B}$. So $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{R})$.

Also, $\mathcal{R} \subseteq \mathcal{C}$, so $\sigma(\mathcal{R}) \subseteq \mathcal{C}$.

(b) if each open subset of Z is a countable union of open rectangles, then each open subset of Z belongs to $\sigma(\mathcal{R})$.

So $\mathcal{C} \subseteq \sigma(\mathcal{R})$.

(c) Suppose X is second countable. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open base for X. Let W be an open subset of Z. Let's show that W is a union of countably many open rectangles.

$\forall n$, let $\mathcal{V}_n = \{V \text{ open} \subseteq Y : U_n \times V \subseteq W\}$. Let $V_n = \bigcup \mathcal{V}_n$.

Then $U_n \times V_n \subseteq W$. Hence, $\bigcup_n (U_n \times V_n) \subseteq W$. In fact, $\bigcup_n (U_n \times V_n) = W$.

Let $(x,y) \in W$. Then $(x,y) \in U \times V \subseteq W$ for some open $U \subseteq X$ & $V \subseteq Y$.

Then $x \in U$ so $x \in U_n \subseteq U$ for some n. Then $V \in \mathcal{V}_n$ so $(x,y) \in U_n \times V_n$.

Products of lots of mble spaces

Let $((X_i, \mathcal{A}_i))_{i \in I}$ be a family of mble spaces.

Let $X = \prod_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i \ \forall i \in I\}$.

$\pi_i : X \longrightarrow X_i$ is defined by $\pi_i((x_j)_{j \in I}) = x_i$.

The product σ-field on X is $\mathcal{A} = \bigotimes_{i \in I} \mathcal{A}_i = \sigma(\pi_i^{-1}[A] : i \in I, A \in X_i)$.

Note that $\bigotimes_{i \in I} \mathcal{A}_i = \sigma\left(\bigodot_{i \in I} \mathcal{A}_i\right)$, where

$$\bigodot_{i \in I} \mathcal{A}_i = \left\{ \prod_{i \in I} A_i : \begin{array}{l} A_i \in \mathcal{A}_i \text{ for all } i \in I \\ A_i = X_i \text{ for all but finitely many } i \in I \end{array} \right\}.$$

The reason is that $\bigodot_{i \in I} \mathcal{A}_i$ is just the set of finite intersections of elements of $\{\pi_i^{-1}[A] : i \in I, A \in \mathcal{A}\}$.

Theorem: Let $((\mathbb{X}_i, \mathcal{A}_i))_{i\in I}$ be a family of mble spaces.

Let $X_i : \Omega \to \mathbb{X}_i$ for each $i \in I$. Let $\mathbb{X} = \prod_{i\in I} \mathbb{X}_i$, $\mathcal{A} = \bigotimes_{i\in I} \mathcal{A}_i$.

Define $X : \Omega \to \mathbb{X}$ by $X(\omega) = (X_i(\omega))_{i\in I}$.

Then $\underbrace{\sigma(X)}_{\substack{\text{smallest } \sigma\text{-field} \\ \text{on } \Omega \text{ which makes} \\ X \text{ mble.}}} = \sigma\underbrace{\left(\bigcup_{i\in I} \sigma(X_i)\right)}_{\substack{\text{smallest } \sigma\text{-field on } \Omega \text{ which} \\ \text{makes each } X_i \text{ mble}}}$ (where $\sigma(X) = \{X^{-1}[A] : A \in \mathcal{A}\}$).

pf For each $i \in I$, for each $A \in \mathcal{A}_i$,

$$X_i^{-1}[A] = (\pi_i \circ X)^{-1}[A] = X^{-1}\Big[\underbrace{\pi_i^{-1}[A]}_{\text{a generator of } \mathcal{A}}\Big] \in \sigma(X)$$

Hence $\sigma(X_i^{-1}[A] : i \in I, A \in \mathcal{A}_i) \subseteq \sigma(X)$.

in other words, $\mathcal{G} = \sigma\left(\bigcup_{i\in I} \sigma(X_i)\right) \subseteq \sigma(X)$.

Now let's show "$\supseteq$".

Let $\mathcal{B} = \{B \subseteq \mathbb{X} : X^{-1}[B] \in \mathcal{G}\}$. Let $i \in I$ and $A \in \mathcal{A}_i$.

Consider $B = \pi_i^{-1}[A]$.

Then $X^{-1}[B] = X^{-1}[\pi_i^{-1}[A]] = (\pi_i \circ X)^{-1}[A] = X_i^{-1}[A] \in \mathcal{G}$.

So $\{\pi_i^{-1}[A] : i \in I, A \in \mathcal{A}_i\} \subseteq \mathcal{B}$, and $\mathcal{B}$ is a σ-field

So $\mathcal{A} \subseteq \mathcal{B}$. So $\forall B \in \mathcal{A}$, $X^{-1}[B] \in \mathcal{G}$, so $\sigma(X) \subseteq \mathcal{G}$.