2/22

Friday, February 22, 2019 14:36

$(\Omega, \mathcal{F}, P)$ - Prob sp.

<u>Defn</u> Let $(\mathcal{H}_i)_{i\in I}$ be a family of π-systems $\mathcal{H}_i \subseteq \mathcal{F}$. To say that $(\mathcal{H}_i)$ is independent means $\forall$ finite $\emptyset \neq I_0 \subseteq I$,

$$\forall (H_i)_{i\in I_0} \in \prod_{i\in I_0} \mathcal{H}_i, \quad P\left(\bigcap_{i\in I_0} H_i\right) = \prod_{i\in I_0} P(H_i).$$

<u>Lemma</u>: Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be indp π-systems. Let $k \in \{1, \dots, n\}$. Then $\mathcal{H}_1, \dots, \sigma(\mathcal{H}_k), \dots, \mathcal{H}_n$ is also indp.

Pf Let $\overset{\emptyset}{\underset{\shortparallel}{}} I_0 \subseteq I$ be finite. Let $(H_i)_{i\in I_0} \in \prod_{i\in I_0} \mathcal{H}_i'$ where $\mathcal{H}_i' = \mathcal{H}_i$ if $i=k$ and $\mathcal{H}_k' = \sigma(\mathcal{H}_k)$. Suppose $k \in I_0$ & $\{k\} \neq I_0$.

Let $I_1 = I_0 \setminus \{k\}$. Let $A = \bigcap_{i\in I_0} H_i$, so $A \cap H_k = \bigcap_{i\in I} H_i$.

Let $\mathcal{B} = \{B \in \mathcal{F} : P(AB) = P(A)P(B)\}$. Then $\mathcal{B}$ is a λ-system on Ω.

Since $\mathcal{H}_1, \dots, \mathcal{H}_n$ are indp, $\mathcal{H}_k \subseteq \mathcal{B}$. Hence $\sigma(\mathcal{H}_k) \subseteq \mathcal{B}$ by the π-λ theorem. In particular, $H_k \in \mathcal{B}$ so $P(\bigcap_{i\in I_0} H_i) = P(A)P(H_k)$. $\square$

<u>Theorem</u>: Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be indp π-systems. Then $\sigma(\mathcal{H}_1), \dots, \sigma(\mathcal{H}_n)$ is are independence.

<u>Theorem</u>: Let $(\mathcal{H}_i)_{i\in I}$ be indp π-systems. Then $(\sigma(\mathcal{H}_i))_{i\in I}$ are indp. π-systems.

<u>Defns</u>

(a) Let $(\mathbb{X}, \mathcal{A})$ be a mble space, and let $X : \Omega \to \mathbb{X}$ be a random variable (i.e. X is $\mathcal{F}/\mathcal{A}$ mble).

Then $\sigma(X) = \{X^{-1}(A) : A \in \mathcal{A}\}$.

(b) Let $((\mathbb{X}_i, \mathcal{A}_i))_{i\in I}$ be a family of mble spaces, and $\forall i \in I$ let $X_i : \Omega \to \mathbb{X}_i$ be a random variable.

To say that the family $(X_i)_{i\in I}$ is <u>independent</u> means that the family $(\sigma(X_i))_{i\in I}$ of σ-fields is independent.

<u>Lemma</u>: Let $(\mathcal{H}_i)_{i\in I}$ be a family of π-systems.

Let $\mathcal{H} = \{\bigcap_{i\in I_0} H_i : \emptyset \neq I_0 \text{ finite} \subseteq I\}$.

Let $\mathcal{G} = \bigcup_{i\in I} \mathcal{H}_i = \{H : H \in \mathcal{H}_i \text{ for some } i \in I\}$.

Then $\mathcal{H} = \pi(\mathcal{G})$.

pf $\forall i$, $\mathcal{H}_i \subseteq \mathcal{H}$ because $\forall H \in \mathcal{H}_i$, $H = \bigcap_{i\in I_0} H_i$ where $I_0 = \{i\}$ and $H_0 = H$.

Hence $\mathcal{G} \subseteq \mathcal{H}$. By construction, each π-system containing $\mathcal{G}$ contains $\mathcal{H}$. It remains only to show that $\mathcal{H}$ is a π-system.

Let $A, B \in \mathcal{H}$. Then $A = \bigcap_{i\in J} A_i$ and $B = \bigcap_{i\in K} B_i$ for some nonempty finite $J, K \subseteq I$ and some $(A_i) \in \prod_{i\in J} \mathcal{H}_i$ and $(B_i) \in \prod_{i\in K} \mathcal{H}_i$.

Then $A \cap B = \bigcap_{i\in J\cup K} A_i \cap B_i$, let's say $B_i = \emptyset\ \forall i \notin K$ and $A_i = \emptyset\ \forall i \in J$. $\square$

The General Grouping Theorem

Let $(\mathcal{H}_i)$

$)_{i \in I}$ be an independent family of π-systems.

Let $(I_j)_{j \in}$ be a disjoint family of nonempty subsets of I.

For each $j \in J$, let $\mathcal{E}_j = \sigma(H : H \in \mathcal{H}_i$ for some $i \in J)$.

Then the family $(\mathcal{E}_j)_{j \in J}$ is independent.

Pf Let $\mathcal{D}_j = \{ \bigcap_{i \in I^\circ} H_i : \emptyset \neq I^\circ \text{ finite} \subseteq I_j, \ H_i \in \mathcal{H}_i \ \forall i \in I^\circ \}$ for each $j \in J$.

Then $\forall j \in J$, $\forall i \in I_j$, $\mathcal{H}_i \subseteq \mathcal{D}_j \subseteq \mathcal{E}_j$. Hence, $\forall j \in J$, $\sigma(\mathcal{D}_j) = \mathcal{E}_j$.

But by the lemma, each $\mathcal{D}_j$ is a π-system.

Hence, by an earlier theorem, to prove that the family $(\mathcal{E}_j)_{j \in J}$ of σ-fields is independent, it suffices to show that $(\mathcal{D}_j)_{j \in J}$ is independent.

Let $\emptyset \neq J^\circ$ finite $\subseteq J$, and let $D_j \in \mathcal{D}_j$ for each $j \in J$.

We wish to show that $P(\bigcap_{j \in J^\circ} D_j) = \prod_{j \in J^\circ} P(D_j)$.

For each $j \in J^\circ$, $D_j = \bigcap_{i \in I_j^\circ} H_i$ for some finite $\emptyset \neq I_j^\circ \subseteq I_j$.

And $\forall i \in I_j^\circ$, $H_i \in \mathcal{H}_i$. (Remember the sets I_j are disjoint

So $(H_i)_{i \in \bigcup_{j \in J^\circ} I_j^\circ}$ is well-defined). Let $I^\circ = \bigcup_{j \in J^\circ} I_j^\circ$. Then $\emptyset \neq I^\circ$ finite $\subseteq I$.

So $P(\bigcap_{j \in J^\circ} D_j) = P(\bigcap_{i \in I^\circ} H_i) = \prod_{i \in I^\circ} P(H_i) = \prod_{j \in J^\circ} P(D_j)$.

eg Let X_1, X_2, X_3, X_4, X_5 be indp real RVs.

Let $Y_1 = f_1(X_1, X_5)$, $Y_2 = f_2(X_2, X_4)$, $Y_3 = f_3(X_3)$.

Where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ and $f_3 : \mathbb{R} \to \mathbb{R}$ are borel fns.

Then Y_1, Y_2, Y_3 are independent RVs.