The monotone class theorem

Let X be a set, let $\mathcal{H} \subseteq \mathcal{P}(X)$, let $\mathcal{M} = MC(\mathcal{H})$. Then:

(a) $\mathcal{M}$ is closed under countable unions iff $\forall A, B \in \mathcal{H}$, $A \cup B \in \mathcal{M}$.

[Pf ($\Rightarrow$) is obvious since $\mathcal{H} \subseteq \mathcal{M}$.

($\Leftarrow$) Suppose $A \cup B \in \mathcal{M}$ $\forall A, B \in \mathcal{H}$.

Let $A \in \mathcal{H}$. Let $\mathcal{B} = \{B \in \mathcal{M} : B \cup A \in \mathcal{M}\}$.

Then $\mathcal{H} \subseteq \mathcal{B}$. Also, $\mathcal{B}$ is a monotone class

(let $(B_n) \in \mathcal{B}^{\mathbb{N}}$. If (B_n) is increasing and $\bigcup_n B_n = B$, then $(A \cup B_n)$ is increasing & $\mathcal{M} \ni \bigcup_n (A \cup B_n) = A \cup (\bigcup_n B_n)$, so $\bigcup_n B_n \in \mathcal{B}$. If (B_n) is decreasing and $\bigcap_n B_n = B$, then $(A \cap B_n)$ is decreasing and $A \cup (\bigcap_n B_n) = \bigcap_n (A \cup B_n) \in \mathcal{M}$. So $\bigcap_n B_n \in \mathcal{B}$. This shows $\mathcal{B}$ is a monotone class.).

Since $\mathcal{B}$ is a monotone class and $\mathcal{H} \subseteq \mathcal{B}$, we have $\mathcal{M} \subseteq \mathcal{B}$. so $\mathcal{B} = \mathcal{M}$ This holds $\forall A \in \mathcal{H}$.

Let $B \in \mathcal{M}$. Let $\mathcal{A} = \{A \in \mathcal{M} : A \cup B \in \mathcal{M}\}$.

Then $\mathcal{H} \subseteq \mathcal{A}$ and $\mathcal{A}$ is a monotone class (by the same argument).

Hence $\mathcal{M} \subseteq \mathcal{A}$. Thus $\forall A, B \in \mathcal{M}$ we have $A \cup B \in \mathcal{M}$.

Now consider any $(A_n) \in \mathcal{M}^{\mathbb{N}}$. Then $\left(B_n = \bigcup_{k \leq n} A_k\right)$ is an increasing sequence so $\bigcup B_n = \bigcup A_n \in \mathcal{M}$]

$(\bigcup_{k\le n} A_k)$ is

an increasing sequence so $\bigcup_n B_n = \bigcup_n A_n \in \mathcal{M}$.]

(b) $\mathcal{M}$ is closed under ctble intersections iff

$\forall A, B \in \mathcal{H}$ we have $A \cap B \in \mathcal{M}$.

[Pf similar to (a).]

(c) $\mathcal{M}$ is closed under complementation rel. to X iff $\forall H \in \mathcal{H}$, $X \setminus H \in \mathcal{M}$.

[Pf ($\Rightarrow$) obvious since $\mathcal{H} \subseteq \mathcal{M}$

($\Leftarrow$) Suppose $X \setminus H \in \mathcal{M}$ $\forall H \in \mathcal{H}$.

Let $\mathcal{A} = \{A \in \mathcal{M} : X \setminus A \in \mathcal{M}\}$. Then $\mathcal{H} \subseteq \mathcal{A}$, and

$\mathcal{A}$ is a monotone class. So $\mathcal{M} = \mathcal{A}$.]

(d) If $\mathcal{H}$ is a field on X, $\mathcal{M}$ is a σ-field on X, $\mathcal{M} = \sigma(\mathcal{H})$.

[Pf $\sigma(\mathcal{H}) \supseteq \mathcal{M}$ since $\sigma(\mathcal{H})$ is a monotone class. But

by the above parts, if $\mathcal{H}$ is a field, $\mathcal{M}$ is a σ-field

containing $\mathcal{H}$ so $\sigma(\mathcal{H}) \subseteq \mathcal{M}$.]

<u>The π-λ theorem</u>: Let X be a set, let $\mathcal{H}$ be a π-system on X.

Let $\mathcal{B}$ be a λ-system on X containing $\mathcal{H}$. Then $\sigma(\mathcal{H}) \subseteq \mathcal{B}$.

Pf Since $\mathcal{H}$ is a π-system and $\mathcal{B}$ is a λ_0-system, $\text{field}(\mathcal{H}) \subseteq \mathcal{B}$.

Since $\mathcal{B}$ is a λ-system, $\mathcal{B}$ is a monotone class, so

$$\mathcal{B} \supseteq MC(\text{field}(\mathcal{H})) = \sigma(\text{field}(\mathcal{H})) = \sigma(\mathcal{H}).$$

$\square$

<u>Exam</u>: Monotone Convergence or <u>a part</u> of Monotone Class Theorem.

Theorem: Let μ and ν be Borel probability measures on $\mathbb{R}$. Let F & G be their Cumulative Distribution Functions. (eg $F(x) = \mu((-\infty, x])$ and $G(x) = \nu((-\infty, x])$). Suppose $F = G$. Then $\mu = \nu$.

pf Let $\mathcal{H} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{H}$ is a π-system. Let $\mathcal{B} = \{B \in \text{Borel}(\mathbb{R}) : \mu(B) = \nu(B)\}$. Since μ and ν are measures on $\text{Borel}(\mathbb{R})$ w/ $\mu(\mathbb{R}) = 1 = \nu(\mathbb{R})$, $\mathcal{B}$ is a λ-system.

$\forall x \in \mathbb{R}$, $\mu((-\infty, x]) = F(x) = G(x) = \nu((-\infty, x])$, so $(-\infty, x] \in \mathcal{B}$.

Thus $\mathcal{H} \subseteq \mathcal{B}$. Therefore $\mathcal{B} \underset{\uparrow \, \pi\text{-}\lambda \text{ theorem}}{\supseteq} \sigma(\mathcal{H}) = \text{Borel}(\mathbb{R})$ so $\mu = \nu$. $\square$

Theorem: Let $(X, \mathcal{A})$ be a mble space, let $\mathcal{H}$ be a π-system on X s.t. $\sigma(\mathcal{H}) = \mathcal{A}$. Let μ and ν be finite measures on $\mathcal{A}$ s.t. $\mu(X) = \nu(X)$. Suppose $\mu(H) = \nu(H)$ $\forall$ $H \in \mathcal{H}$. Then $\mu = \nu$.

Theorem: Let $(X, \mathcal{A})$ be a mble space and let $\mathcal{H}$ be a π-system on X s.t. $\sigma(\mathcal{H}) = \mathcal{A}$. Let μ and ν be measures on $\mathcal{A}$ s.t. $\mu(H) = \nu(H)$ $\forall$ $H \in \mathcal{H}$. Suppose $\exists$ $(H_n) \in \mathcal{H}^{\mathbb{N}}$ s.t. $\bigcup_n H_n = X$ and $\mu(H_n) = \nu(H_n) < \infty$.

pf $\forall n \in \mathbb{N}$, define μ_n and ν_n on $\mathcal{A}$ by $\mu_n(A) = \mu(A \cap H_n)$, $\nu_n(A) = \nu(A \cap H_n)$. Then $\mu_n(X) = \mu(H_n) = \nu(H_n) = \nu_n(X) < \infty$, so μ_n and ν_n are finite measures on $\mathcal{A}$ with the same total mass.

For each $H \in \mathcal{H}$, we have $H \cap H_n \in \mathcal{H}$ so $\mu_n(H) = \mu(H \cap H_n) = \nu(H \cap H_n) = \nu_n(H)$. Thus by the above result, $\nu_n = \mu_n$. This holds $\forall n$.

Let $A \in \mathcal{A}$. Then $A = \bigcup_n (A \cap H_n)$. Let $A_n = A \cap H_n$. Let $B_n = A_n \setminus \bigcup_{m<n} A_m$.

Then $A = \dot{\bigcup_n} B_n$. Also, $\forall n$, $\mu(B_n) = \mu(B_n \cap H_n) = \mu_n(B_n)$, same for ν so:

$$\mu(A) = \sum_n \mu(B_n) = \sum_n \mu_n(B_n) = \sum_n \nu_n(B_n) = \sum_n \nu(B_n) = \nu(A) \qquad \square$$

<u>Theorem</u>: Let μ and ν be measures on $\mathrm{Borel}(\mathbb{R})$ s.t. $\mu((a,b]) = \nu((a,b]) < \infty$ for $-\infty < a \leq b < \infty$. Then $\mu = \nu$.

Pf Take $\mathcal{H} = \{(a,b] : -\infty < a \leq b < \infty\}$, take $H_n = (-n, n]$. $\square$