π- systems

$$\pi(A_1,\dots,A_n) = \left\{ \bigcap_{i\in I} A_i : \emptyset \neq I \subseteq \{1,\dots,n\} \right\}.$$

Defn: Let X be a set. To say $\mathcal{A}$ is a λ-system on X means

(a) $X \in \mathcal{A}$,

(b) $\forall A \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$

(c) $\forall$ disjoint $(A_n) \in \mathcal{A}^{\mathbb{N}}$, $\bigcup_n A_n \in \mathcal{A}_n$

Remarks: . σ-fields are λ-systems, but not conversely
(λ-system union requires disjoint)

Defn A λ_0-system on X is like a λ-system but
we only get finite unions.

Example: Let $(X,\mathcal{A})$ be a mble space. let μ & ν be
finite measures on $\mathcal{A}$ with $\mu(X) = \nu(X)$.
let $\mathcal{B} = \{ B \in \mathcal{A} : \mu(B) = \nu(B) \}$.

Then $\mathcal{B}$ is a λ-system on X.

Example: Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Let $A \in \mathcal{F}$. Let $\mathcal{B} = \{B \in \mathcal{F} : A \text{ and } B \text{ are independent}\}$.

Then $\mathcal{B}$ is a λ-system on Ω.

Pf We'll deduce this from the previous example.

Define μ and ν on $\mathcal{F}$ by $\mu(B) = P(AB)$ and $\nu(B) = P(A)P(B)$.

Then μ and ν are measures on $\mathcal{F}$ with $\mu(\Omega) = P(A\Omega) = P(A)$,

$\nu(\Omega) = P(A) \cdot 1 = P(A)$.

So $\mu(\Omega) = \nu(\Omega) < \infty$.

And $\mathcal{B} = \{B \in \mathcal{A} : \mu(B) = \nu(B)\}$, so $\mathcal{B}$ is a λ-system.

Propn: Let $\mathcal{B}$ be a λ_0-system. Let $A, B \in \mathcal{B}$ with $A \subseteq B$.

Then $B \setminus A \in \mathcal{B}$ as well.

Pf $B \setminus A = X \setminus (A \sqcup (X \setminus B))$.

Propn: Let $\mathcal{B}$ be a λ_0-system on X. Let $A, B \in \mathcal{B}$.

Then TFAE:

(a) $A \cap B \in \mathcal{B}$

(b) $A \setminus B \in \mathcal{B}$

(c) $A \cup B \in \mathcal{B}$

(c) $A \cup B \in \mathcal{B}$

(d) $B \setminus A \in \mathcal{B}$

Pf (a) $\Rightarrow$ (b) : $A \setminus B = A \setminus (A \cap B)$

(b) $\Rightarrow$ (c) : $A \cup B = (A \setminus B) \sqcup B$

(c) $\Rightarrow$ (d) : $B \setminus A = (A \cup B) \setminus A$

(d) $\Rightarrow$ (a) : $A \cap B = B \setminus (B \setminus A)$

Propn Let $\mathcal{B}$ be a λ-system on X. Let $A_1, \dots, A_n \in \mathcal{B}$ s.t. $\pi(A_1, \dots, A_n) \subseteq \mathcal{B}$. Let $k \in \{1, \dots, n\}$. Let $B_i = A_i$ if $i \neq k$, and $B_k = A_k^c$. Then $\pi(B_1, \dots, B_n) \subseteq \mathcal{B}$ as well.

Pf Let $\emptyset \neq I \subseteq \{1, \dots, n\}$. We wish to show that $\bigcap_{i \in I} B_i \in \mathcal{B}$.

This is obvious if $k \notin I$ or if $I = \{k\}$.

Suppose $k \in I$ and $I \neq \{k\}$. Let $J = I \setminus \{k\}$. Then

$\bigcap_{i \in I} B_i = A_k^c \cap \left(\bigcap_{j \in J} A_j\right) \in \mathcal{B}$ by the previous propn, because

$\bigcap_{j \in J} A_j \in \mathcal{B}$ and $A_k \in \mathcal{B}$, $\left(\bigcap_{j \in J} A_j\right) \cap A_k = \bigcap_{i \in I} A_i \in \mathcal{B}$,

and $\left(\bigcap_{j \in J} A_j\right) \cap A_k^c = \left(\bigcap_{i \in I} A_i\right) \setminus A_k$.

Corollary: Let $\mathcal{B}$ be a λ-system on a set X.

Let $A_1, \dots, A_n \in \mathcal{B}$ such that $\pi(A_1, \dots, A_n) \subseteq \mathcal{B}$.

Then:

(a) for each $(B_i) \in \prod_{i=1}^{n} \{A_i, A_i^c\}$, $\pi(B_1, \dots, B_n) \subseteq \mathcal{B}$.

(b) atoms $(A_1, \dots, A_n) \subseteq \mathcal{B}$.

(c) field $(A_1, \dots, A_n) \subseteq \mathcal{B}$.

the "π-λ_0 theorem"

<u>Corollary</u>: Let $\mathcal{B}$ be a λ_0-system on X. Let $\mathcal{H}$ be a π-system on X such that $\mathcal{H} \subseteq \mathcal{B}$. Then field $(\mathcal{H}) \subseteq \mathcal{B}$

Pf by the previous corollary $\forall$ finite $\mathcal{H}_0 \subseteq \mathcal{H}$, we have field $(\mathcal{H}_0) \subseteq \mathcal{B}$, because $\pi(\mathcal{H}_0) \subseteq \mathcal{H} \subseteq \mathcal{B}$. But field $(\mathcal{H}) = \bigcup_{\mathcal{H}_0 \subseteq \mathcal{H}} \text{field}(\mathcal{H}_0) \subseteq \mathcal{B}$.

<u>Defn</u>: A monotone class is a collection of sets that is closed under countable increasing unions and countable decreasing intersections.

<u>Example</u>: a σ-field.

a λ-system

Pf obvious: $\bigcup A_n = \bigcup (A_{n+1} \setminus A_n)$ if $A_n \subseteq A_{n+1}$ ...

Also, if $A_n \supseteq A_{n+1}$ then $\bigcap A_n = X \setminus \bigcup_n (X \setminus A_n)$

<u>Propn</u> Given any set $\mathcal{H}$ of sets, there is a smallest monotone class $MC(\mathcal{H})$ containing $\mathcal{H}$.

Pf Let $X = \cup \mathcal{H}$. Let $\Gamma = \{\mathcal{M} \subseteq P(X) : \mathcal{M}$ is a monotone class containing $\mathcal{H}\}$. Let $\mathcal{M}_0 = \cap \Gamma$. Then $MC(\mathcal{H}) = \mathcal{M}_0$.