Reminder Let $A_1, \dots, A_n \subseteq \Omega$.

Let $\mathcal{B} = \{B \subseteq \Omega : B \text{ is expressible in terms of } A_1, \dots, A_n\}$

Define $X : \Omega \longrightarrow \{0, 1\}^n$ by $X(\omega) = (1_{A_1}(\omega), \dots, 1_{A_n}(\omega))$.

Let $\mathbb{X} = X[\Omega]$. Then:

(a) $C \longmapsto X^{-1}[C]$ is a bijection from $\mathcal{P}(\mathbb{X})$ onto $\mathcal{B}$.

(b) $|\mathcal{B}| = 2^m$ where $m = |\mathbb{X}| \leq 2^n$.

Further Remarks

(c) for each $x = (x_1, \dots, x_n) \in \{0,1\}^n$,

$$X^{-1}[\{x\}] = \{\omega \in \Omega : X(\omega) = x\}$$

$$= \{\omega \in \Omega : 1_{A_j}(\omega) = x_j \text{ for } j = 1, \dots, n\}$$

$$= \bigcap_{j=1}^{n} \{\omega \in \Omega : 1_{A_j}(\omega) = x_j\}$$

$$= \bigcap_{j=1}^{n} B_j \quad \text{where} \quad B_j = A_j \text{ if } x_j = 1$$
$$= A_j^c \text{ if } x_j = 0.$$

note that $\bigcap_{j=1}^{n} B_j \neq \emptyset$ iff $X^{-1}[\{x\}] \neq \emptyset$

iff $x \in X[\Omega]$

$$\text{atoms}(A_1, \dots, A_n) = \left\{ E : \emptyset = E = \bigcap_{j=1}^{n} B_j \text{ for some } (B_j) \in \prod_{j=1}^{n} \{A_j, A_j^c\} \right\}$$

$$= \{ X^{-1}[\{x\}] : x \in \mathbb{X} \}$$

(d) Since $\mathcal{B} = \{ X^{-1}[C] : C \subseteq \mathbb{X} \}$, and since $\mathcal{P}(\mathbb{X})$ is a field of subsets of $\mathbb{X}$, $\mathcal{B}$ is a field of subsets of Ω.

for $j = 1, \dots, n$, A_j is expressible in terms of $A_1, \dots, A_n$

so $A_j \in \mathcal{B}$. also, $A_j = X^{-1}[C_j]$ where $C_j = \{ \overset{(x_1,\dots,x_n) \in \mathbb{X}}{\overset{\shortparallel}{x}} : x_j = 1 \}$.

$\mathcal{B}$ is a field on Ω containing $A_1, \dots, A_n$.

Suppose $\mathcal{B}'$ is any field on Ω containing $A_1, \dots, A_n$. Then $\mathcal{B}' \supseteq \mathcal{B}$.

Thus $\mathcal{B}$ is the smallest field on Ω containing $A_1, \dots, A_n$.

$\mathcal{B} = \text{field}(A_1, \dots, A_n)$

if we apply this procedure to an infinite family

$(A_j)_{j \in J}$, we get the $\underbrace{\text{complete field}}_{\text{arbitrary unions}}$ generated by (A_j).

Prop Let Γ be a nonempty $\underbrace{\text{upward directed}}$ set of fields on Ω.

$\forall \mathcal{B}_1, \mathcal{B}_2 \in \Gamma$, $\exists \mathcal{B}_3 \in \Gamma$ s.t. $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}_2$.

Let $\mathcal{C} = \cup \Gamma = \{c : c \in \mathcal{B} \text{ for some } \mathcal{B} \in \Gamma\}$.

Then $\mathcal{C}$ is a field on Ω

pf Let $C_1, C_2 \in \mathcal{C}$. $C_1 \in \mathcal{B}_1$, $C_2 \in \mathcal{B}_2$, so $\exists\, \mathcal{B}_3 \supseteq \mathcal{B}_1, \mathcal{B}_2$, so $C_1 \cup C_2 \in \mathcal{B}_3 \subseteq \mathcal{C}$.

Theorem: Let $\mathcal{H}$ be any set of subsets of Ω.

Then $\text{field}(\mathcal{H}) = \cup \underbrace{\{\text{field}(H_1, \ldots, H_n) : n \geq 0,\ H_1, \ldots, H_n \in \mathcal{H}\}}_{\text{upwards directed}}$

pf obvious, given preceeding discussion

eg Let $\Omega = \mathbb{Z}$ and $\mathcal{H} = \{\{n\} : n \in \mathbb{Z}\}$.

then $\text{field}(\mathcal{H}) = \{A \subseteq \mathbb{Z} : A \text{ is finite or } \mathbb{Z} \setminus A \text{ is finite}\}$.

$\sigma(\mathcal{H}) = \mathcal{P}(\mathbb{Z})$

eg Let $\Omega = \mathbb{R}$ and $\mathcal{H} = \{\{x\} : x \in \mathbb{R}\}$. Then $\mathrm{field}(\mathcal{H}) = \{A \subseteq \mathbb{R} : A \text{ is finite or } \overbrace{\mathbb{R} \setminus A \text{ is finite}}^{\text{cofinity}}\}$.

$\sigma(\mathcal{H}) = \{A \subseteq \mathbb{R} : A \text{ is ctble or co-ctble}\}$

eg Let $\Omega = \mathbb{R}$ and $\mathcal{H} = \{(-\infty, r) : r \in \mathbb{Q}\}$.

$$\mathrm{field}(\mathcal{H}) = \left\{ \dot{\bigcup_{j=0}^{n}} I_j : \begin{array}{l} I_0 \text{ is of the form } (-\infty, b_0) \text{ or is empty} \\ n \in \mathbb{N} \quad I_n \text{ is of the form } [a_n, \infty) \text{ or is empty} \\ I_j \text{ is of the form } [a_j, b_j) \text{ or is empty for } 0 < j < n \end{array} \right\}$$

$\sigma(\mathcal{H}) = \mathrm{Borel}(\mathbb{R})$

complete field $= \mathcal{P}(\mathbb{R})$

Uniqueness of measures, etc.

Let X be a set.

A π-system on X is a set $\mathcal{H}$ of subsystems of X which is closed under finite intersections.

eg Let $\mathcal{H} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$

Then $\mathcal{H}$ is a π-system.

eg Let $\mathcal{H} = \{ \prod_{i=1}^{d} (a_i, b_i] : a_i, b_i \in \mathbb{R},\ a_i \le b_i\ \forall i \}$ is also a π-system.

eg Let X be a set and let $A_1, \dots, A_n \subseteq X$. Then

$$\underbrace{\pi(A_1, \dots, A_n)}_{\text{Smallest } \pi\text{-system containing } A_1, \dots, A_n.} = \left\{ \bigcap_{i \in I} A_i : \emptyset \neq I \subseteq \{1, \dots, n\} \right\}$$