<u>Thm</u> Let $f, g : X \longrightarrow (-\infty, \infty]$ be mble. Suppose $\int f\, d\mu$ and $\int g\, d\mu$ are defined and $> -\infty$. Then $\int f+g\, d\mu$ is defined and equals $\int f d\mu + \int g\, d\mu$.

<u>Pf</u> Let $h_1 = f^+ + g^+$ and $h_2 = f^- + g^-$. Then $h_1 : X \longrightarrow [0, \infty]$ and $h_2 : X \to [X, \infty)$ are mble and $\int h_2 d\mu = \int f^- d\mu + \int g^- d\mu < \infty$ (since both are less than ∞ bc $\int f d\mu, \int g d\mu > -\infty$).

$f + g = (f^+ - f^-) + (g^+ - g^-) = (f^+ + g^+) - (f^- + g^-) = h_1 - h_2$.

$\int h_1 - h_2\, d\mu = \int h_1 d\mu - \int h_2 d\mu$ by lemma from last time.

$$= \int f^+ + g^+ d\mu - \int f^- + g^-\, d\mu$$

$$= \int f^+ d\mu + \int g^+ d\mu - \int f^- d\mu - \int g^-\, d\mu$$

$$= \int f\, d\mu + \int g\, d\mu \qquad \square$$

Now back to $\mathbb{R}^d$:

$(X, \mathcal{A}, \mu)$ a measure space.

if $f : X \longrightarrow \mathbb{R}^d$ then $f(x) = (f_1(x), \ldots, f_d(x))$ $\forall x \in X$ where $f_1, \ldots, f_d : X \longrightarrow \mathbb{R}$.

from last time, f is mble iff $f_1, \ldots, f_d$ are mble.

<u>Defn</u> (a) f is μ-$\int$ble iff each $f_1, \ldots, f_d$ is μ-$\int$ble.

(b) if f is μ-$\int$ble then $\int f d\mu = \left(\int f_1 d\mu, \ldots, \int f_d d\mu\right)$.

Propn: f is μ-$\int$ble iff f is mble & $\int |f| d\mu < \infty$.

Pf we know f is mble iff $f_1, \ldots, f_d$ are all mble. $\forall x \in X$,

$$|f(x)| = \left|\sum_{j=1}^{d} f_j(x)\, u_j\right| \le \sum_{j=1}^{d} |f_j(x)|.$$

If f is μ-$\int$ble then $f_1, \ldots, f_d$ are all μ-$\int$ble so $\int |f_j| d\mu < \infty$ $\forall j$.

So $\int |f| d\mu < \infty$.

Conversely, if $\int |f| d\mu < \infty$ then $\forall j$, $\forall x \in X$,

$$|f_j(x)| = \sqrt{f_j(x)^2} \le \sqrt{\sum_{k=1}^{d} f_k(x)^2} = |f(x)| \text{ so } \int |f_j| d\mu < \infty \text{ so } f_j \text{ is } \mu\text{-}\textstyle\int\text{ble}. \quad \square$$

Propn: Let $f: X \longrightarrow \mathbb{R}^d$ be μ-$\int$ble. Then $\left|\int f d\mu\right| \le \int |f| d\mu$.

Pf Let $y = \int f d\mu$ if $y = 0$ there is nothing to show.

if not, let $u = \frac{y}{|y|}$. Then $|y| = \frac{\langle y, y \rangle}{|y|} = \langle u, y \rangle = \langle u \mid \int f d\mu \rangle$

$$= \sum_{j=1}^{d} u_j \int f_j d\mu = \int \sum_{j=1}^{d} u_j f_j(x)\, d\mu(x) = \int \langle u, f(x) \rangle\, d\mu(x)$$

Cauchy-Schwarz inequality $\longrightarrow$ $\le \int |u| \cdot |f(x)|\, d\mu(x) = \int |f| d\mu$. $\square$

The Dominated Convergence Theorem

Let $f, f_1, f_2, f_3, \ldots : X \longrightarrow \mathbb{R}^d$ be mble. Let $g: X \to [0, \infty]$ be mble with $\int g\, d\mu < \infty$.

Suppose $\forall x$ that $|f_n(x)| \le g(x)$ $\forall n$ and $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$.

Then

(a) : $\int |f - f_n| \, d\mu \longrightarrow 0$ as $n \longrightarrow \infty$.

(b) : $\int f_n \, d\mu \longrightarrow \int f \, d\mu$ as $n \longrightarrow \infty$.

Proof: (a) is already done.

(b) $\forall x \in X$, $|f(x)| \le g(x)$, so $|f - f_n| \le |f| + |f_n| \le 2g$, So $f - f_n$ is μ-$\int$ble.

$$\left| \int f d\mu - \int f_n d\mu \right| = \left| \int f - f_n \, d\mu \right| \le \int |f - f_n| \, d\mu \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Back to probability theory:

$(\Omega, \mathcal{F}, P)$ a probability space.

If $(\mathbb{X}, \mathcal{A})$ is a mble space, a random variable in $\mathbb{X}$ is an $\mathcal{F}/\mathcal{A}$-measurable fn $X : \Omega \longrightarrow \mathbb{X}$.

A real-valued RV is an $\mathcal{F}/\mathcal{B}$-measurable fn $X : \Omega \longrightarrow \mathbb{R}$ where $\mathcal{B} = \text{Borel}(\mathbb{R})$.

If X is a real-valued RV and $\int |X| \, dP < \infty$, then $\int X \, dP$ is called the expected value of X, denoted $E(X)$.

Functions expressable in terms of other functions:

Let X and Y be fns on Ω. To say Y is expressible in terms of X means $\forall \omega_1, \omega_2 \in \Omega$, if $X(\omega_1) = X(\omega_2)$ then $Y(\omega_1) = Y(\omega_2)$.

If f is a function w/ domain containing $\mathrm{Rng}(X)$ and $Y = f \circ X$, then Y is expressible in terms of X.

Conversely, suppose Y is expressible in terms of X. Let $f : \mathrm{Rng}(X) \longrightarrow \mathrm{Rng}(Y)$ be given by $f = \{(X(\omega), Y(\omega)) : \omega \in \Omega\}$. Then $Y = f \circ X$.

Let $A_1, \dots, A_n, B \subseteq \Omega$. To say B is expressible in terms of $A_1, \dots, A_n$ means 1_B is expressible in terms of $(1_{A_1}, \dots, 1_{A_n})$.

Propn Let $B \subseteq \Omega$ and let $X : \Omega \longrightarrow \mathbb{X}$. Then 1_B is expressible in terms of X iff $B = X^{-1}(C)$ for some $C \subseteq \mathbb{X}$.

Pf: ($\Rightarrow$) Suppose 1_B is expressible in terms of X. Then $1_B = f \circ X$ for some fn f on $\mathrm{Rng}(X)$.
$f(X(\omega)) = 1$ if $\omega \in B$, $f(X(\omega)) = 0$ if $\omega \notin B$. Let $C = f^{-1}(1)$.
Then $B = X^{-1}(C)$.

($\Leftarrow$) Let $f(x) = 1_C$ so $B = f \circ X$.

Corollary Let $A_1, \dots, A_n, B \subseteq \Omega$. B is expressible in terms of $A_1, \dots, A_n$ iff $B = X^{-1}(C)$ for some $C \subseteq \{0,1\}^n$ where $X : \Omega \to \{0,1\}^n$ is $X(\omega) = (1_{A_1}(\omega), \dots, 1_{A_n}(\omega))$.

Corollary Let $A_1, \dots, A_n, \subseteq \Omega$. Let $\mathbb{X} = \{X(\omega) : \omega \in \Omega\}$ ("Range(X)") where X is as above.
Let $\mathcal{B} = \{B \subseteq \Omega : B$ is expressible in terms of $A_1, \dots, A_n\}$, Let $\mathcal{C} = \mathcal{P}(\mathbb{X})$
then the map $C \overset{(*)}{\longmapsto} X^{-1}[C]$ is a bijection $\mathcal{C} \longrightarrow \mathcal{B}$.

Pf if B is expressible in terms of $A_1, \dots, A_n$ then $\exists\, C \in \mathcal{C}$ s.t. $B = X^{-1}[C]$, and conversely (by above corollary). Range$((*)) = \mathcal{B}$.
Now let $C \in \mathcal{C}$. Then $X[X^{-1}[C]] = C \cap \mathrm{Range}(X) = C$.
So X is one-to-one as well. □

Corollary Let $A_1, \ldots, A_n$, X, and $\mathbb{X}$ be as in the preceding corollary. Let $m = |\mathbb{X}|$.

Then $m \le 2^n$ and $|\mathcal{B}| = 2^m$. So $|\mathcal{B}| \le 2^{2^n}$.