$(X, \mathcal{A}, \mu)$ a m.ble space.

<u>Defn</u> Let $\varphi : X \longrightarrow [0, \infty)$ be simple.

Then $$\int_X \varphi \, d\mu = \sum_{y \in [0,\infty)} y \cdot \mu(\varphi = y)$$ (this is a finite sum).

Convention: $0 \cdot \infty = 0$.

<u>Lemma</u>: Let $\varphi : X \longrightarrow [0, \infty)$ be simple.

Suppose $X = \dot{\bigcup}_{k=1}^{n} A_k$ where $A_1, \dots, A_n \in \mathcal{A}$.

Then $$\int \varphi \, d\mu = \sum_{k=1}^{n} \underbrace{\int_{A_k} \varphi \, d\mu}_{\int 1_{A_k} \cdot \varphi \, d\mu}.$$

<u>Propn</u>: Let $\varphi, \psi : X \longrightarrow [0, \infty)$ be simple.

Then $$\int \varphi + \psi \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu.$$

<u>Pf</u> $$\int \varphi + \psi \, d\mu = \sum_w w \cdot \mu(\varphi + \psi = w) = \sum_w w \cdot \sum_{u,v} \mu(\varphi + \psi = u, \ \varphi = u, \psi = v).$$

$$= \sum_{u,v} \sum w \cdot \mu(\varphi + \psi = w, \ \varphi = u, \ \psi = v) = \sum_{u,v} (u + v) \cdot \mu(\varphi = u, \ \psi = v)$$

$$= \sum_{u,v} u \cdot \mu(\varphi = u, \psi = v) + \sum_{u,v} v \cdot \mu(\varphi = u, \psi = v)$$

$$= \sum_u u \cdot \sum_v \mu(\varphi = u, \psi = v) + \sum_v v \cdot \sum_u \mu(\varphi = u, \psi = v)$$

$$= \sum_u u \cdot \sum_v \mu(\varphi = u, \psi = v) + \sum_v v \cdot \sum_u \mu(\varphi = u, \psi = v)$$

$$= \sum_u u \cdot \mu(\varphi = u) + \sum_v v \cdot \mu(\psi = v)$$

$$= \int \varphi \, d\mu + \int \psi \, d\mu$$

Remark we used $\mu(A) = \sum_y \mu(A \cap \{f = y\})$
if f is simple.

Defn: Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be mble spaces.
Let $f : X \longrightarrow Y$. To say f is mble (or $\mathcal{A}/\mathcal{B}$-mble)
if $\forall\ B \in \mathcal{B}$, $f^{-1}[B] \in \mathcal{B}$.

Propn: Suppose $\mathcal{B} = \sigma(\mathcal{H})$ for some family $\mathcal{H}$ of
Subsets of Y. f is mble iff $f[H] \in \mathcal{A}\ \forall\ H \in \mathcal{H}$.

Pf. $\Rightarrow$ obvious: $H \in \mathcal{H} \Rightarrow H \in \mathcal{B}$.
$\Leftarrow$ Suppose $\forall\, H \in \mathcal{H}$, $f^{-1}[H] \in \mathcal{A}$.
Let $\mathcal{C} = \{C \subseteq Y : f^{-1}[C] \in \mathcal{A}\}$.
Then $\mathcal{H} \subseteq \mathcal{C}$.
Now $Y \in \mathcal{C}$ since $f^{-1}[Y] = X \in \mathcal{A}$.

$\forall C \in \mathcal{C}$, $f^{-1}[Y \setminus C] = X \setminus f^{-1}[C] \in \mathcal{A}$.

if $C_1, C_2, \ldots \in \mathcal{C}$, then

$$f^{-1}\left[\bigcup_{n=1}^{\infty} C_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[C_n] \in \mathcal{A}.$$

So $\mathcal{C}$ is a σ-field on Y containing $\mathcal{H}$

So $\mathcal{C} \supseteq \sigma(\mathcal{H}) = \mathcal{B}$.

Thus f is mble.

eg Let $f : \underset{\mathcal{A}}{X} \longrightarrow \underset{\mathcal{B} = \mathrm{Borel}(\mathbb{R})}{\mathbb{R}}$.

Then f is mble iff for each $y \in \mathbb{R}$, $\{f > y\} \in \mathcal{A}$.

eg $(X, \mathcal{A})$ a mble space. Y a topological space.

Let $\mathcal{B} = \mathrm{Borel}(Y)$ and $f : X \to Y$.

f is $\mathcal{A}/\mathcal{B}$ measurable if $f^{-1}[U] \in \mathcal{A}$ $\forall$ openset $U \subseteq Y$.

eg X, Y topological spaces. $\mathcal{A} = \mathrm{Borel}(X)$, $\mathcal{B} = \mathrm{Borel}(Y)$.

$f : X \xrightarrow{\text{cts}} Y$ is $\mathcal{A}/\mathcal{B}$-mble.

Propn: Let $f_1, f_2, f_3, \ldots$ be mble fns on $(X, \mathcal{A})$

taking values in $\overline{\mathbb{R}} = [-\infty, \infty]$.

Then the following are mble functions:

(a) $\sup\limits_n f_n$

(b) $\inf\limits_n f_n$

(c) $\limsup\limits_{n\to\infty} f_n$

(d) $\liminf\limits_{n\to\infty} f_n$

(e) $\lim\limits_{n\to\infty} f_n$, if it exists $\forall x \in X$.

Pf (a) $\{\sup_n f_n > y\} = \{x \in X : \sup_n f_n(x) > y\}$

$$= \{x \in X : \exists n \; f_n(x) > y\}$$

$$= \bigcup_n \{x \in X : f_n(x) > y\}$$

$$= \bigcup_n \underbrace{\{f_n > y\}}_{\in \mathcal{A}} \in \mathcal{A}$$

$\forall y \in \mathbb{R}$, so $\sup_n f_n$ is mble.

(b) is (a) upside down

(c) $$\limsup_{n\to\infty} f_n = \underbrace{\inf_n \underbrace{\left(\sup_{m\geq n} f_n\right)}_{\text{mble by a}}}_{\text{mble by b}}$$

(d) is (c) inside out.

(e) is (c) & (d) when $\lim_{n\to\infty} f_n(x)$ exists $\forall x \in X$.

Simple fns are mble, so limits of simple fns are mble.

Propn: Let $f : X \longrightarrow [0,\infty]$ be mble. Then $\exists$ an increasing sequence (φ_n) of simple fns $\varphi_n : X \longrightarrow [0,\infty)$ s.t. $\varphi_n \uparrow f$ pointwise on X.

Pf: Let (y_k) be a dense sequence in $[0,\infty)$ (eg an enumeration of $\mathbb{Q}_{\geq 0}$).

Let $A_k = \{f \geq y_k\}$. Let $\psi_k = y_k 1_{A_k}$, and

$\varphi_n = \max\{\psi_1, \dots, \psi_n\}$.

$$\psi_k = \begin{cases} 0 & \text{on } X \setminus A_k \\ y_k & \text{on } A_k = \{f \geq y_k\} \end{cases}$$

So $\psi_k \leq f$ so $\varphi_n \leq f$.

Clearly $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots$

Let $x \in X$. Suppose $0 \leq y < f(x)$.

Then $(y, f(x))$ is a non-empty open $\subseteq [0,\infty)$.

So $\exists$ k s.t. $y_k \in (y, f(x))$, so $\forall n \geq k$, $\varphi_n(x) \geq y_k > y$

Thus $\varphi_n(x) \longrightarrow f(x)$ as $n \to \infty$.

If $f(x) = 0$ then $\varphi_n(x) = 0 \ \forall n$ so it still works.