$\int$ is additive for simple fns satisfying conditions.

More about σ-fields

① There is a smallest σ-field $\mathcal{B}$ on $\mathbb{R}$ containing all intervals.

Let $\mathcal{G} = \{G \subseteq \mathbb{R} : G \text{ is open}\}$

and $\mathcal{F} = \{F \subseteq \mathbb{R} : F \text{ is closed}\}$

Then $\mathcal{G}_\delta = \{\bigcap_{n=1}^{\infty} G_n : \text{each } G_n \in \mathcal{G}\} \subseteq \mathcal{B}$

eg: $[0,1] = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, 1+\frac{1}{n}) \in \mathcal{G}_\delta$

$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\}) \in \mathcal{G}_\delta$

eg $\mathbb{Q} \notin \mathcal{G}_\delta$

(if $\mathbb{Q} \in \mathcal{G}_\delta$, Then $\emptyset = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ would be a countable intersection of dense open subsets of $\mathbb{R}$. but a countable intersection of dense open subsets of $\mathbb{R}$ is dense in $\mathbb{R}$, by the Baire Category Theorem.)

$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \in \mathcal{F}_\sigma$

$$\ldots = \bigcup_{q\in\mathbb{Q}} \ldots \in \mathcal{G}_\sigma$$

$$\overset{\mathcal{H}_0}{\overset{\shortparallel}{\mathcal{G}}} \subsetneq \overset{\mathcal{H}_1}{\overset{\shortparallel}{\mathcal{G}_\delta}} \subsetneq \overset{\mathcal{H}_2}{\overset{\shortparallel}{\mathcal{G}_{\delta\sigma}}} \subsetneq \overset{\mathcal{H}_3}{\overset{\shortparallel}{\mathcal{G}_{\delta\sigma\delta}}} \subsetneq \overset{\mathcal{H}_4}{\overset{\shortparallel}{\mathcal{G}_{\delta\sigma\delta\sigma}}} \subsetneq \cdots$$

All of these are subsets of $\mathcal{B}$.

But there are sets in $\mathcal{B}$ which aren't in any of these.

Let $\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_n$.

$\mathcal{H}_\delta, \mathcal{H}_{\delta\sigma}, \mathcal{H}_{\delta\sigma\delta}, \ldots$

$\mathcal{B}$ contains all of these and more.

$\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_\omega, \mathcal{H}_{\omega+1}, \ldots, \mathcal{H}_{\omega+\omega}, \mathcal{H}_{\omega+\omega+1}, \ldots$

$$\mathcal{B} = \bigcup_{\alpha < \aleph_1} \mathcal{H}_\alpha.$$

$\hookrightarrow$ or Ω, the first uncountable ordinal.

<u>Theorem</u>: Let X be a set & let $\mathcal{E}$ be any set of subsets of X. Then $\exists$ a smallest σ-field $\mathcal{A}$ on X with $\mathcal{E} \subseteq \mathcal{A}$.

<u>Proof</u>: Let $\mathcal{A}$ be the set of all $A \subseteq X$ s.t. for each

σ-field $\mathcal{B}$ on X, if $\mathcal{E} \subseteq \mathcal{B}$, Then $A \in \mathcal{B}$.

(i.e. $\mathcal{A}$ = intersection of all σ-fields containing $\mathcal{E}$).

↳ not empty bc $\mathcal{P}(X)$ is a σ-field.

Then ① $\mathcal{E} \subseteq \mathcal{A}$

② $\mathcal{A}$ is a σ-field on X

③ for each σ-field $\mathcal{B}$ on X, if $\mathcal{E} \subseteq \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{B}$. □

We write $\mathcal{A} = \sigma(\mathcal{E})$.

↑ depends on X too.

Propn: Let $\mathcal{E}_1, \mathcal{E}_2$ be sets of subsets of X. Then

(a) $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2) \iff \mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$.

(b) $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) \iff \mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$.

Defn: if X is a topological space, the borel σ-field on X is

$$\text{Borel}(X) = \sigma(\{U \subseteq X : U \text{ is open}\}).$$

eg: Let $\mathcal{H} = \{(a,b] : -\infty < a \leq b < \infty\}$.

$$\sigma(\mathcal{H}) = \text{Borel}(\mathbb{R})$$

pf: $\text{Borel}(\mathbb{R}) = \sigma(\mathcal{G})$.

$(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$ so $\mathcal{H} \subseteq \sigma(\mathcal{G})$, so $\sigma(\mathcal{H}) \subseteq \sigma(\mathcal{G})$.

Let $G \in \mathcal{G}$. Let $\mathcal{U} = \{(a,b] : a, b \in \mathbb{Q},\ a<b,\ (a,b] \subseteq G\}$.

Let $U = \bigcup \mathcal{U}$. Then $U \subseteq G$. And if $x \in G$ then

$x \in (x-\delta, x+\delta) \subseteq G$ for some $\delta > 0$.

choose rational #s $a \in (x-\delta, x)$ and $b \in (x, x+\delta)$.

Then $x \in (a,b] \subseteq G$. So $x \in U$. So $U = G$.

So $\mathcal{G} \subseteq \sigma(\mathcal{H})$ so $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{H})$.

So $\sigma(\mathcal{H}) = \text{Borel}(\mathbb{R})$.

Each of the following collections generates Borel$(\mathbb{R})$ too:

1 $\{(a,\infty) : a \in \mathbb{R}\}$, $\{(-\infty, b) : b \in \mathbb{R}\}$

2 $\{[a,\infty) : a \in \mathbb{R}\}$, $\{(-\infty, b] : b \in \mathbb{R}\}$

eg let $\Sigma = \{0,1\}^{\mathbb{N}}$

for each $n \in \mathbb{N}$, for each $s = (s_1, \dots, s_n) \in \{0,1\}^n$,

Let $\Sigma(s) = \{(t_1, t_2, \dots) \in \Sigma : t_i = s_i\ \forall i \le n\}$.

The natural σ-field on Σ is $\sigma(\{\Sigma(s) : n \in \mathbb{N},\ s \in \{0,1\}^n\})$.

This is Borel (Σ).

<u>Fact</u>: There is a unique measure m° on $\mathrm{Borel}(\mathbb{R})$ s.t. $\forall a, b \in \mathbb{R}$, if $a \le b$, $m^\circ((a,b]) = b - a$.

m° is called Borel-Lebesgue measure on $\mathbb{R}$.

eg There is a unique measure P on $\mathrm{Borel}(\Sigma)$ s.t. for each $n \in \mathbb{N}$ and for each $s \in \{0,1\}^n$, $P(\Sigma(s)) = \frac{1}{2^n}$.