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- n men put their hats in a sack.
- each man draws a hat & puts it on.

P(at least one man gets his own hat back) = ?

for $i = 1, \dots, n$, let A_i = event that i^{th} man gets his own hat back.

Then $P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} A_{i_2}) + \sum_{i_1 < i_2 < i_3} P(A_{i_1} A_{i_2} A_{i_3}) - \cdots$$

$$+ (-1)^{n-1} \sum_{\substack{i_1 < \cdots < i_n \\ (1 < \cdots < n)}} \underbrace{P(A_{i_1} \cdots A_{i_n})}_{\text{actually } P(A_1 \cdots A_n)}$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i_1 < i_2} \frac{(n-2)!}{n!} + \sum_{i_1 < i_2 < i_3} \frac{(n-3)!}{n!} - \cdots + (-1)^{n-1} \frac{1}{n!}$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \longrightarrow 1 - \frac{1}{e}$$

A proof of PIE

For any event B, $E(1_B) = P(B)$

For any bounded RVs $X_1, \dots, X_n$,

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$$

Let $A_1, \dots, A_n$ be events.

$$P(A_1 \cup \cdots \cup A_n) = E(1_{A_1 \cup \cdots \cup A_n}) .$$

Now

$$1_{A_1 \cup \cdots \cup A_n} = 1 - 1_{A_1^c \cap \cdots \cap A_n^c} = 1 - 1_{A_1^c} \cdot \cdots \cdot 1_{A_n^c}$$

$$= 1 - (1 - 1_{A_1}) \cdots (1 - 1_{A_n})$$

$$= 1 - \sum_{I \subseteq [n]} \prod_{i \in I} (-1_{A_i})$$

$$= \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} 1_{\bigcap_{i \in I} A_i}$$

So $$P(A_1 \cup \cdots \cup A_n) = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} P\Big(\bigcap_{i \in I} A_i\Big)$$

<u>Simple functions.</u>

Let $(X, \mathcal{A})$ be a mble space

X: a set, $\mathcal{A}$: a σ-algebra on X

Let Y be a set. A simple function from X to Y is a function $\varphi : X \longrightarrow Y$ s.t. $\varphi[X]$ is finite and $\forall y \in \varphi[X]$, $\varphi^{-1}[\{y\}] \in \mathcal{A}$. $\left(\varphi^{-1}[\{y\}] = \{x \in X : \varphi(x) = y\} = \{\varphi = y\} \right)$.

Propn: Let $\varphi : X \longrightarrow Y$ be simple. Let $B \subseteq Y$. Then $\varphi^{-1}[B] \in \mathcal{A}$.

Pf $B = \bigcup_{y \in B} \{y\}$, so $\varphi^{-1}[B] = \bigcup_{y \in B} \varphi^{-1}[\{y\}]$, which is a finite union of sets in $\mathcal{A}$, so it's in $\mathcal{A}$.

Propn: Let $\varphi : X \longrightarrow Y$ be simple, and let $h : Y \longrightarrow Z$. Then $h \circ \varphi$ is simple.

Pf Obvious. (but it does use the above propn).

Propn: Let $\varphi_k : X \longrightarrow Y_k$ for $k = 1, \dots, n$. Let $Y = Y_1 \times \cdots \times Y_n$.

Define $\varphi : X \longrightarrow Y$ by $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

Then φ is simple iff $\varphi_1, \dots, \varphi_n$ are simple.

Pf: Suppose φ is simple. $\varphi_k = \pi_k \circ \varphi$ for all k.

Suppose $\varphi_1, \dots, \varphi_n$ are simple Then φ is obviously simple