Friday, January 11, 2019 14:25

a family $(x_\alpha)_{\alpha \in A}$ is a function x whose domain is A and $x(\alpha) = x_\alpha$.

$\{ x_\alpha : \alpha \in A \}$ is range(x).

$(x_\alpha)_{\alpha \in A} = \{ (\alpha, x_\alpha) : \alpha \in A \}$.

A choice function for a family of sets $(X_\alpha)_{\alpha \in A}$ is a family $(x_\alpha)_{\alpha \in A}$ s.t. $\forall \alpha \in A$, $x_\alpha \in X_\alpha$.

If $(X_\alpha)_{\alpha \in A}$ is a family of sets, then

$\prod_{\alpha \in A} X_\alpha$ = the set of all choice functions for $(X_\alpha)_{\alpha \in A}$

$= \{ (x_\alpha)_{\alpha \in A} : \text{for each } \alpha \in A,\ x_\alpha \in X_\alpha \}$.

The axiom of choice says that each family of non-empty sets has at least one choice function.

$$Y^X = \{ f : X \to Y \} = \prod_{x \in X} Y$$

$$f : \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]$$

$$f((x_n)_{n \in \mathbb{N}}) = \sum \frac{x_n}{2^n}$$

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$n \in \mathbb{N}$

$$\operatorname{rng}(f) = [0,1]$$

f isn't 1-1. But it nearly is. $E = \left\{ \frac{m}{2^n} \,.\, n \in \mathbb{N},\ m \in \{1,2,\dots,2^n-1\} \right\}$.

Then $\forall x \in [0,1]$, $\left| f^{-1}(\{x\}) \right| = \begin{cases} 1 & x \in [0,1] \setminus E \\ 2 & x \in E \end{cases}$.

Hence $\left| \{0,1\}^{\mathbb{N}} \right| = \left| [0,1] \right|$. $\approx$ equinumerous.

$$[0,1] \times [0,1] \approx \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \underset{\substack{\uparrow \\ \text{interleave}}}{\approx} \{0,1\}^{\mathbb{N}} \approx [0,1]$$

$$[0,1]^{\mathbb{N}} \approx \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \text{ "=" } \{0,1\}^{\mathbb{N}\times\mathbb{N}} \underset{\substack{\uparrow \\ \text{dovetail}}}{\approx} \{0,1\}^{\mathbb{N}} \approx [0,1].$$

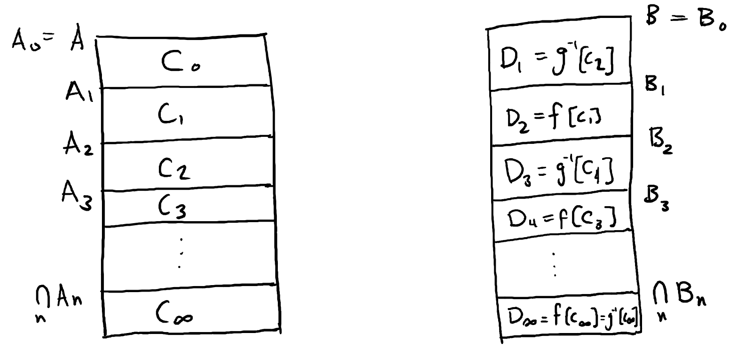
$$[0,1] \approx \{0,1\}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}} \preceq [0,1]^{\mathbb{N}} \approx [0,1]$$

$\Rightarrow [0,1] \approx \mathbb{N}^{\mathbb{N}}$, by the

Schroeder-Bernstein Theorem: $A \preceq B$ & $B \preceq A \Rightarrow A \approx B$.

Pf Let $f : A \longrightarrow B$ & $g : B \longrightarrow A$ be injections.

$$A_i = g(B_{i-1}) \qquad B_i = f(A_{i-1})$$



<u>let</u> $h(x) = \begin{cases} f(x) & x \in C_{2n-1} \\ g^{-1}(x) & x \in C_{2n} \\ f(x) & x \in C_\infty \end{cases}$ for some $n \in \mathbb{N}$

Then h is a bijection from A to B.