Lec 8/29

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How many permutators are needed to represent a given bistochastic matrix?

$$\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} = \frac{1}{n!} \sum \text{all permutators}$$

$$= \frac{1}{n} \sum n \text{ permutators.}$$

hint: vicinity of n^2

Exercise: find a tight universal upper bound for this (in terms of n)

$$P = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & & 0 \\ 1 & & & \end{pmatrix}. \quad \{I, P, P^2, \dots, P^{n-1}\} \text{ is a cyclic group.}$$

Ex: give all examples of groups w/ no nontrivial subgroups

$$\det\left((a_{ij})_{i,j=1}^n\right) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n a_{i\sigma(i)}.$$

no sign of σ

Permanent: $\operatorname{per}\left((a_{ij})_{i,j=1}^n\right) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$

Remember this one.

in 20s or 30s.

↳ Van Der Waerden's Permanent Conjecture: The minimum permanent of a bistochastic matrix is $\frac{n!}{n^n}$.

Answer: yes. Falikman & Egorychev 1970s. Check in Proofs from THE BOOK

$P_n \sim n \log n$, $\pi(n) \sim \frac{n}{\log n}$

Ex. find order of magnitude of $\pi(n)$, assuming $P_n \sim n \log n$

$a_n \sim n^2$. $\#\{a_n : a_n \le m\} \overset{??}{\sim} \sqrt{n}$

Check proof of Stirling's formula

Check Wikipedia article for Double-Counting and the one for inclusion-exclusion

ex: geometric proof of $1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

ex: prove $\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$

ex: Prove $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

ex: Prove $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

ex: Prove any permutation is a product of disjoint cycles.

Defn: order of $x \in G$.

ex: how many cyclic permutations are there in S_n.

ex: what is the order of a product of disjoint cycles

eigenvalues of $\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$ are n^{th} roots of unity.

Ex. $\exists T$ s.t. $T\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_{n-1} \end{pmatrix}$ (show it is normal $\Rightarrow$ diagonalizable).

Ex: What is the proportion in $\mathbb{N}$ of those n which can be written as sums of distinct powers of 3.