Math 5529H In-Class Exercises

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Week 1

1.1 Prove that there is only one 3×3 magic square using each of the numbers from 1 to 9 once.

1.2: Let A, B be $n \times n$ matrices. Show that you need (practically) the same number of multiplication operations to compute AB, A^{-1}, and $\det A$.

1.3: Can the number which a magic square's rows/columns/diagonals add up to ever be a prime?

1.4: Use the definition of the kth fibonacci number F_k, which is $F_0 = 0, F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$, to prove that $\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}} = 1$.

1.5: Consider the game of tossing a coin (with sides A and B) and recording the results until two As appear in a row. For example, one event in this probabilistic game is $BABAA$, which has length 5. Show that the number of events with length n is the fibonacci number F_{n-1} where $F_0 = 0, F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$.

1.6: Prove that the set of infinite-length events possible in the game described above is uncountable. Justify the equation $\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}} = 1$ probabilistically, given that fact.

1.7: Generalize the above exercises to a coin with probability p of landing on side A.

1.8: Characterize those $x \in [0, 1)$ with two n-ary expansions $x = \sum_{k=1}^{\infty} \frac{a_k}{n^k}$ (each $a_k \in \{0, 1, \ldots, n-1\}$).

1.9: Show that a finite algebra $\mathscr{A}$ of subsets of some set X satisfies $|\mathscr{A}| = 2^k$ for some $k \in \mathbb{N}$.

1.10: Prove that the utilities problem is impossible to solve (i.e. prove that $K_{3,3}$ is nonplanar)

Week 2

2.1: Let G be a finite group. Show that there is some $n \in \mathbb{N}$ so that G is a subgroup of S_n, the group of permutations on n elements.

2.2: Find a tight upper bound (in terms of n) for the maximum number of permutators needed in a convex combination to represent an arbitrary bistochastic matrix of size $n \times n$.

2.3: List all groups with no nontrivial subgroups.

2.4: Let p_n be the nth prime and let $\pi(n)$ be the number of primes $\leq n$. Given that $p_n \sim n \log n$, show that $\pi(n) \sim \frac{n}{\log n}$.

2.5: Prove that $\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$ (without using Stirling's formula).

2.6: Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

2.7: Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

2.8: How many cyclic permutations are there in S_n?

2.9: Prove that any permutation in S_n is a product of disjoint cycles.

2.10: What is the order of a product of disjoint cycles?

2.11: Let $P = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$ be an $n \times n$ bistochastic matrix. Show that there is some $n \times n$ invertible matrix T so that TPT^{-1} is a diagonal matrix whose diagonal entries are the nth roots of unity (i.e. show that P has the nth roots of unity as its eigenvalues).

2.12: What is the proportion in $\mathbb{N}$ of those n which can be written as sums of distinct powers of 3?

2.13: Let $\mathscr{F}$ be a collection of sets with $|\mathscr{F}| = n$. Show that the algebra generated by $\mathscr{F}$ has at most 2^{2^n} sets.

2.14: Find a formula (or at least an estimate) for the sum of the first n elements in the kth diagonal of Pascal's triangle (where the 0th diagonal is $1, 1, 1, 1, \ldots$, the first diagonal is $1, 2, 3, 4, \ldots$, the second diagonal is $1, 3, 6, 10, \ldots$, et cetera).

2.15: Show that any natural number can be represented as a sum of distinct Fibonacci numbers, and this representation is unique as long as we disallow two consecutive Fibonacci numbers to be used.

2.16: Show that if a sequence $(a_n) \subset \mathbb{N}$ (with $a_n \to \infty$ as $n \to \infty$) has slower growth than 2^n then there will be some integer which has multiple representations as a sum of distinct a_n's.

2.17: Define $d_p, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by $d_p(x, y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}$ for each $p \in [0, \infty)$ and $d_\infty(x, y) = \max \{|x_i - y_i|\}_{i=1}^{n}$ (where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$). Show that for any $x, y \in \mathbb{R}^n$, we have $\lim_{p \to \infty} d_p(x, y) = d_\infty(x, y)$.

Week 3

3.1: Give an example of a finite sum set $FS(x_n) := \{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k, k \in \mathbb{N}\}$ (where (x_n) is a sequence in $\mathbb{N}$ with $x_n \to \infty$ as $n \to \infty$) such that there are no arithmetic triples (x, y, z satisfying $x + y = 2z$) in $FS(x_n)$.

3.2: Give an example of a non-linear equation which is solvable in $FS(x_n)$ for every (x_n).

3.3: Show the following three forms of van der Waerden's theorem are equivalent:
 (a) $\mathbb{N} = \bigcup_{i=1}^r C_i \Rightarrow$ some C_i is AP-rich.
 (b) If $S \subset \mathbb{N}$ is AP-rich and $S = \bigcup_{i=1}^r C_i$, then some C_i is AP-rich.
 (c) $\forall \ell, r \in \mathbb{N}$, $\exists N \in \mathbb{N}$ so that if $\{1, 2, \ldots, N\} = \bigcup_{i=1}^r C_i$ then some C_i contains an AP of length ℓ.

3.4: The Erdős-Szekeres theorem is: for all $n \in \mathbb{N}$, there is some $N \in \mathbb{N}$ so that any set of N points in general position in $\mathbb{R}^2$ contains an empty convex n-gon (here "empty" means that the n-gon of points does not contain any of the other points in its interior).

A version of Ramsey's Theorem is: Denote by $S^{(2)}$ the set of all 2-element subsets of S. Let S be a countably infinite set. for any finite coloring $S = \bigcup_{i=1}^r C_i$, some C_i contains a set of the form $R^{(2)}$, where $R \subset S$ is infinite.

Use this version of Ramsey's Theorem to prove the Erdős-Szekeres theorem.

3.5: Let $FS(x_n)$ denote the set of finite sums of distinct elements of the infinite sequence (in $\mathbb{N}$) (x_n). Let $FP(x_n)$ denote the set of finite products of distinct elements of (x_n). Prove that the following to versions of Hindman's Theorem are equivalent:
 (a) $FS(x_n) = \bigcup_{i=1}^r C_i \Rightarrow$ some C_i contains $FS(y_n)$ for some (y_n).
 (b) $FP(x_n) = \bigcup_{i=1}^r C_i \Rightarrow$ some C_i contains $FP(y_n)$ for some (y_n).

3.6: Using van der Waerden's theorem, prove that for any finite coloring of $\mathbb{N}^2$, any infinite line of lattice points $\{a + tb : t \in \mathbb{Z}\}$ (where $a, b \in \mathbb{Z}^2$) contains arbitrarily long arithmetic progressions in one color.

3.7: Using van der Waerden's theorem, prove that for any finite coloring of $\mathbb{N}^2$, one color contains arbitrarily large grids of the form $\{a, a + d_1, \ldots, a + (n-1)d_1\} \times \{b, b + d_2, \ldots, b + (n-1)d_2\}$.

3.8: Formulate a version of van der Waerden's theorem for $\mathbb{N}^2$ (which is stronger than van der Waerden's theorem in $\mathbb{N}$) in three different ways.

3.9: Invent a Ramsey-type theorem about complete graphs.

3.10: Let V be an infinite vector space over $\mathbb{Z}/p\mathbb{Z}$. Invent a Ramsey-type theorem about such a V.

Radical Pi Exercise: Let $\mathrm{Conv}(A)$ be the convex hull of A. Show that the following three theorems are equivalent:
 (a) If $A_1, \ldots, A_k$ are convex sets in $\mathbb{R}^n$ such that any $n+1$ of them intersect, then all of them intersect.
 (b) For any set A of $n+2$ points in $\mathbb{R}^n$, there is a partition $A = A_1 \sqcup A_2$ so that $\mathrm{Conv}(A_1) \cap \mathrm{Conv}(A_2) \neq \varnothing$.
 (c) For any $S \subset \mathbb{R}^n$, any point in $\mathrm{Conv}(S)$ is in $\mathrm{Conv}(S_1)$ for some $S_1 \subset S$ with $|S_1| \leq n+1$.

Week 4

4.1: Recall that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$ (these are Fibonacci numbers) and that this matrix is diagonalizable: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = A^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n A$ (these λ's are the eigenvalues of the matrix). Calculate A and use this information to show that $F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$.

4.2: Find a relationship between the recurrence $V_{n+3} = V_{n+2} + V_{n+1} + V_n$ and the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

4.3: Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) = cx$ for some $c \in \mathbb{R}$.

4.4: Given a countable set D, create a monotone function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous precisely on D.

4.5: Assume Champernowne's constant 0.1234567891011121314151617... is normal in base 10. Prove the corresponding number in base 2 (which has binary expansion 0.11011100101110111...) is normal in base 2.

4.6: If $A_i \subset \mathbb{R}$ with $\mu(A_i) = 0$ for $i \in \mathbb{N}$, show that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

4.7: Show that the classical middle-thirds Cantor set has measure 0.

4.8: Prove that $[0,1]$ does not have measure 0.

4.9: Show that there is some $S \subset \mathbb{R}$ so that $S \cap I$ is uncountable for any interval $I \subset \mathbb{R}$ and $\mu(S) = 0$.

4.10: Show that the set of non-normal numbers (in base 2) is uncountable.

4.11: Let C be the classical middle-thirds Cantor set. Show that $C + C = [0,2]$ and $C - C = [-1,1]$ (recall that $A + B = \{a + b : a \in A, b \in B\}$ and similarly for $A - B$).

4.12: Let C be the classical (middle thirds) cantor set. Is it true that $FS(C) = [0, \infty)$?

4.13: If f is a function $\{1, \ldots, n\} \to \{1, \ldots, n\}$, let $a_k^{(f)} = \left|\left\{ f^k(1), \ldots, f^k(n) \right\}\right|$ (where f^k is f applied iteratively k times). Show that $\left| \left\{ \left(a_n^{(f)} \right)_{n=1}^{\infty} : f \in [n]^{[n]} \right\} \right| = 1 + p(1) + p(2) + \cdots + p(n-1)$, where $[n]^{[n]}$ is the set of functions from $[n] := \{1, \ldots, n\}$ to itself and p is the partition function.

4.14: Consider the sequence of first digits of 2^n: $1, 2, 4, 8, 1, 3, 6, \ldots$. Which number appears more frequently: 7 or 8?

4.15: Does pairwise independence (of sets in a probability space) imply joint independence?

Week 5

5.1: Show that for any $t \in \mathbb{Z}$ and any $E \subset \mathbb{N}$, $\bar{d}(E-t) = \bar{d}(E)$ (where $x \in E-t$ iff $x+t \in E$).

5.2: Show that if $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ then at least one C_i has $\bar{d}(C_i) \geq \frac{1}{r}$.

5.3: Using Szemerédi's theorem, show that if $E \subset \mathbb{N}^2$ with $\bar{d}(E) > 0$ then E contains arbitrarily large patterns of the form $\{a, a+d_1, a+2d_1 \ldots, a+(n-1)d_1\} \times \{b, b+d_2, b+2d_2, \ldots, b+(n-1)d_2\}$. For our purposes, the upper density for $\mathbb{N}^2$ is $\bar{d}(E) = \overline{\lim}_{N\to\infty} \frac{|E\cap\{1,2,\ldots,N\}^2|}{N^2}$.

5.4: Is it true that $A \sqcup (A-1) = \mathbb{N} \Rightarrow d(A) = \frac{1}{2}$?

5.5: For $E \subset \mathbb{N}$, define $\bar{d}_\times(E) = \overline{\lim}_{N\to\infty} \frac{|E\cap F_N|}{|F_N|}$ where $F_N = \left\{2^{i_1}3^{i_2}5^{i_3}\cdots p_N^{i_N} : 0 \leq i_j \leq N \text{ for all } j\right\}$. Show that $\bar{d}_\times(\mathbb{N}) = 1$, $\bar{d}_\times(2\mathbb{N}) = 1$, and $\bar{d}_\times(3\mathbb{N}+1) = 0$.

5.6: Define $E_\times = \left\{n = 2^{c_1}3^{c_2}5^{c_3}\cdots p_k^{c_k} : k \in \mathbb{N}, \sum_{i=1}^{k} c_i \in 2\mathbb{N}\cup\{0\}\right\}$ (the "multiplicatively even" numbers) and define $O_\times = \mathbb{N}\setminus E_\times = \left\{n = 2^{c_1}3^{c_2}5^{c_3}\cdots p_k^{c_k} : k \in \mathbb{N}, \sum_{i=1}^{k} c_i \in 2\mathbb{N}-1\right\}$ (the "multiplicatively odd" numbers). Show that $d_\times(E_x) = d_\times(O_x) = \frac{1}{2}$ (where $d_\times$ is like $\bar{d}_\times$ but with lim instead of $\overline{\lim}$).

5.7: Show that for any $t \in \mathbb{N}$ and any $E \subset \mathbb{N}$, $\bar{d}_\times(E) = \bar{d}_\times\left(\frac{E}{t}\right)$ (where $x \in \frac{E}{t}$ iff $xt \in E$).

5.8: Let X be a set with $|X| = n$. Show that $\mathscr{P}(X)$ (the power set of X) is a 2^n element group under the operation Δ defined by $A\Delta B = (A\cup B)\setminus(A\cap B)$.

5.9: Let $S \subset \mathbb{N}$ and define $M_S = \left\{2^{c_1}3^{c_2}5^{c_3}\cdots p_k^{c_k} : k \in \mathbb{N}, \sum_{i=1}^{k} c_i \in S\right\}$. Is it true that $d_\times(M_S) = d(S)$? What about $\bar{d}_\times(M_S) = \bar{d}(S)$?

5.10: Let $\alpha > 1$ and $S_\alpha = \{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$. Show that $d(S_\alpha) = \frac{1}{\alpha}$.

5.11: Let $\alpha, \beta \notin \mathbb{Q}$ so that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Prove that $S_\alpha \sqcup S_\beta = \mathbb{N}$.

5.12: Show that if $\alpha \notin \mathbb{Q}$ then S_α does not contain an infinite arithmetic progression.

5.13: Let $\mathbb{F}_5$ be the finite field with 5 elements. Note that 2 and 3 are both not squares in $\mathbb{F}_5$. Show that $\left\{a + b\sqrt{2} : a, b \in \mathbb{F}_5\right\}$ and $\left\{a + b\sqrt{3} : a, b \in \mathbb{F}_5\right\}$ are both fields, and that they are isomorphic.

5.14: Create a field with p^2 elements for any prime p (including $p = 2$).

Week 6

6.1: Let (I_n) be a sequence of intervals in $\mathbb{N}$ (sets of consecutive integers) with $|I_n| \to \infty$ as $n \to \infty$. Define $\bar{d}_{(I_n)}(E) = \limsup_{n\to\infty} \frac{|E\cap I_n|}{|I_n|}$. Show that for any (I_n), $E \subset \mathbb{N}$, and $t \in \mathbb{N}$, $\bar{d}_{(I_n)}(E-t) = \bar{d}_{(I_n)}(E)$.

6.2: Let (I_n) be a sequence of intervals in $\mathbb{N}$ with $|I_n| \to \infty$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $\lim_{n\to\infty} \frac{f(n)}{|I_n|} = 0$, and define $J_n = I_n + f(n)$. Show that for any $E \in \mathbb{N}$, $\bar{d}_{(J_n)}(E) = \bar{d}_{(I_n)}(E)$.

6.3: Define "Upper Banach Density" as $d^*(E) = \sup\left\{\bar{d}_{(I_n)}(E) : (I_n) \text{ with } |I_n| \to \infty \text{ as } n \to \infty\right\}$ (where (I_n) always denotes a sequence of intervals in $\mathbb{N}$). Can the sup in this definition be replaced by max?

6.4: Do there exist countably infinitely many disjoint sets $E_1, E_2, \ldots \subset \mathbb{N}$ so that $d^*(E_n)$ for all n?

6.5: Do there exist countably infinitely many disjoint sets $E_1, E_2, \ldots \subset \mathbb{N}$ so that $\bar{d}(E_n)$ for all n?

6.6: Prove that the following forms of Szemerédi's theorem are equivalent:
- (a) $\forall E \subset \mathbb{N},$ if $\bar{d}(E) > 0,$ then E is AP-rich.
- (b) $\forall E \subset \mathbb{N},$ if $d^*(E) > 0,$ then E is AP-rich.
- (c) $\forall \alpha \in (0,1), \forall \ell \in \mathbb{N}, \exists N(\alpha,\ell)$ so that if $M \geq N, E \subset \{1,2,\ldots,M\}$ with $\frac{|E|}{M} \geq \alpha$ then E contains a length-ℓAP.

6.7: Prove that the power set of A, $\mathscr{P}(A)$, satisfies $|\mathscr{P}(A)| > |A|$ for all sets A.

6.8: Prove that $\left(\{0,1\}^{\mathbb{N}}, \rho\right)$ is a compact metric space, where $\rho(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$.

6.9: Show that $C_0 = \{x \in X : x_1 = 0\}$ is a clopen subset of $\{0,1\}^{\mathbb{N}}$.

6.10: Give an example of an open (and not closed) subset of $\{0,1\}^{\mathbb{N}}$.

6.11: Is it true that if (A_i) is a mutually independent family of Borel sets (meaning that we have $\mu\left(\bigcap_{j=1}^{n} A_{i_j}\right) = \prod_{j=1}^{n} \mu\left(A_{i_j}\right)$ for every sequence $(i_j)_{j=1}^{n}$) then (B_i) is too, where each B_i is either A_i or A_i^C?

6.12: Let $V_{\mathbb{F}_p} = \{(a_1, a_2, \ldots) : a_i \in \mathbb{F}_p$ and only finitely many a_i's are nonzero$\}$ be a vector space over $\mathbb{F}_p$, the field with p elements (for some prime p). If $V_{\mathbb{F}_p} = \bigcup_{i=1}^{r} C_i$, is it true that some C_i contains an infinite affine subspace?

Week 7

7.1: Show the following three forms of Hindman's theorem are equivalent:

(a) For any finite coloring of the natural numbers $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one C_i contains a set of the form $FS(x_n)$ for some sequence $(x_n) \subset \mathbb{N}$ with $x_n \nearrow \infty$.

(b) For any (x_n) with $x_n \nearrow \infty$, and for any finite coloring of the finite sum set $FS(x_n) = \bigcup_{i=1}^{r} C_i$, one C_i contains a set of the form $FS(y_n)$ for some other sequence $(y_n) \subset \mathbb{N}$ with $y_n \nearrow \infty$.

(c) Let $\mathscr{F}$ be the set of all nonempty finite subsets of $\mathbb{N}$. If $\mathscr{F} = \bigcup_{i=1}^{r} C_i$ then one C_i contains some set of finite unions, $FU(\alpha_i) = \{\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k} : k \in \mathbb{N}, i_1 < i_2 < \cdots < i_k\}$, where all of the (infinitely many) α_i are disjoint.

Week 8

8.1: Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous bijection so that for every $x \in \mathbb{S}^1$, there is some $n \in \mathbb{N}$ for which $f^n(x) = x$ (here f^n is the nth iterate of f). Is there necessarily some $N \in \mathbb{N}$ so that f^N is the identity function?

8.2: If $E \subset \mathbb{N}$ is a normal set, show that for any k integers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_1 < t_2 < \cdots < t_k$, we have $d\left(E \cap (E - t_1) \cap (E - t_2) \cap \cdots \cap (E - t_k)\right) = \frac{1}{2^{k+1}}$.

8.3: Show that if $\bigcup_{i=1}^{r} (E - t_i) = \mathbb{N}$ (i.e. E is syndetic) then $d^*(E) \geq \frac{1}{r}$.

8.4: Show that if $\bar{d}(E) > 0$ then $E - E$ is syndetic.

8.5: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $S = \{n : 0 < n\alpha \pmod 1 < \epsilon \text{ or } 1 - \epsilon < n\alpha \pmod 1 < 1\}$. Show that S is Δ^*.

8.6: Let $\bar{d}(E) > 0$, and let $n_i \nearrow \infty$. Show that there are indices $i < j$ so that $(E - n_i) \cap (E - n_j) \neq \varnothing$.

8.7: Let E_1, E_2 be Δ^*. Show that $E_1 \cap E_2$ is Δ^*.

Week 9

9.1: Does Van der Waerden's theorem follow from the fact that any syndetic set is AP-rich?

9.2: Show that if $\mathbb{N}\bigcup_{i=1}^{r} C_i$ then one of C_i is piecewise syndetic (the intersection of a syndetic set and a thick set).

9.3: If E is piecewise syndetic and $E = \bigcup_{i=1}^{r} C_i$ then one C_i is piecewise syndetic.

9.4: Let P be the set of primes. Does $P - 1$ contain an IP-set?

9.5: Show that if a group $G \leq V_{\mathbb{F}_p}$ is infinite then $G \cong V_{\mathbb{F}_p}$.

9.6: Show that $V_{\mathbb{F}_p}$ only has countably many finite subgroups.

9.7: Show that any $A \subset \mathbb{N}$ with $\bar{d}(A) = 1$ is GP-rich.

9.8: Is it true that for any $\epsilon > 0$ there is some $A \subset \mathbb{N}$ with $\bar{d}(A) > 1 - \epsilon$ and A is not GP-rich.

9.9: Show that any $A \subset \mathbb{N}$ with $\bar{d}(A) = 1$ is AP-rich (without using Szemerédi's theorem).

9.10: Show that if $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ then $\mathbb{N} = \left(\bigcup_{j=1}^{k} C_{i_j}\right) \cup \left(\bigcup_{j=1}^{\ell} C_{i'_j}\right)$ where $\bar{d}\left(\bigcup_{j=1}^{k} C_{i_j}\right) = 1$ (i.e. we can split up the colors into two groups, one of which has a union with density 1).

9.11: Let E_1, E_2 be IP*. Show that $E_1 \cap E_2$ is IP*.

9.12: $A \subset \mathbb{N}$ is defined to be "multiplicatively thick" if for any finite set F, there is some $x \in \mathbb{N}$ so that $\frac{A}{x} \supset F$. Prove or disprove: an IP* set is multiplicatively thick.

9.13: If $\lim_{n\to\infty} \frac{a_n}{2^n} < 1$ is it true that $\bar{d}(A) > 0$?

9.14: Prove that any multiplicatively thick set contains a multiplicative IP set.

9.15: In a naïve way, define an IP set in a (not necessarily commutative) semigroup S as a set of finite products of some sequence in S (in any order). Show that Hindman's theorem is not true with this definition.

Week 10

10.1: Is it true that if $\mathbb{N} = \bigcup_{i=1}^r C_i$, one C_i contains a finite sums set and a finite products set?

10.2: Is there a set of positive natural density which contains no shift of an IP set?

10.3: Let $A \subset \mathbb{N}$ with $d(A) = \frac{1}{2}$ and $d(A \cap (A - t)) = \frac{1}{4}$ for all $t \in \mathbb{N}$. Is it true that for any $t_1 < t_0 < \cdots < t_n \in \mathbb{N}$, $d(A \cap (A - t_1) \cap \cdots \cap (A - t_n)) = \frac{1}{2^{n+1}}$?

10.4: Let $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$. Show that there is some $n \in \mathbb{N}$ so that $\bar{d}(A \cap (A - n)) > 0$.

10.5: Show that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to $\{0,1,\ldots,r-1\}^{\mathbb{N}}$ (using the usual metric).

10.6: Let $S \subset \mathbb{Z}$ and suppose that there are $a, b, c \in \mathbb{Z}$ so that $(S + a) \sqcup (S + b) \sqcup (S + c) = \mathbb{Z}$. Show that S is periodic (i.e. that there is some $k \in \mathbb{Z}$ so that $S = S - k$).

10.7: Let P be a partition-regular property of subsets of natural numbers so that if A has P and $n \in \mathbb{N}$ then nA has P. Call a set P^* if it intersects any P set. Show that a P^* set is multiplicatively thick.

10.8: Show that there are uncountably many different IP sets in $\mathbb{N}$.

10.9: A minimal subsystem of a topological dynamic system (X, T) (X is a topological space and $T : X \to X$ is a homeomorphism) is a closed subset $Y \subset X$ so that $TY \subset Y$, and Y does not contain a proper subsystem. Show that (X, T) Is minimal if and only if any point has dense orbit.

10.10: Show that the following two formulations of van der Waerden's theorem are equivalent:

(a) If $\mathbb{N} = \bigcup_{i=1}^r C_i$, one C_i is AP-rich.

(b) If X is a compact metric space and $T : X \to X$ is a homeomorphism, for any $k \in \mathbb{N}$ and any $\epsilon > 0$ there is some $n \in \mathbb{N}$ and $x \in X$ so that $\operatorname{diam}\left(\left\{x, T^n x, T^{2n} x, \ldots, T^{kn} x\right\}\right) < \epsilon$.

10.11: Let P be a property of subsets of natural numbers. Let $\mathbf{1}_P : \mathbb{P} \to \{0,1\}$ be the indicator function of P (i.e. if $A \subset \mathbb{N}$ has property P, $\mathbf{1}_P(A) = 1$, and $\mathbf{1}_P(A) = 0$ otherwise). Show that if $\mathbf{1}_P$ is an ultrafilter then $P = P^*$ (recall that $A \subset \mathbb{N}$ is P^* if it intersects any P set). Is the converse true?

10.12: Show that $P^* = P^{***}$ for any property P of subsets of natural numbers.

10.13: Show that $\left\{n : \|n^2\alpha\| < \epsilon\right\}$ is not Δ^* for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$.

10.14: Define syndeticity and thickness in a general semigroup. Show that Syndetic* = Thick and Thick* = Syndetic.

10.15: Let B be a convex body in $\mathbb{R}^2$ (i.e. B is closed, bounded, has nonempty interior, and is convex). Show that any $x \in B$ is a convex combination of 3 boundary points of B.

10.16: Let B be the set of $n \times n$ bistochastic matrices as a subset of $\mathbb{R}^{n^2}$. Show that the extreme points of B (the points which cannot be realized as a nontrivial convex combination of other points in B) are exactly the permutator matrices.

10.17: A line in a matrix is either a column or a diagonal. Show that the minimum number of lines needed to cross out all 1s in a $0-1$ matrix is the same as the maximum size of an independent set in the matrix (a set of entries which are all 1 and appear in different rows and columns).

Week 11

11.1: Can $\mathbb{N} = A \cup M$ with $\bar{d}(A) = 0$ and $\bar{d}_\times(M) = 0$?

11.2: (a) Let $A, B \subset \mathscr{P}(\{1, \dots, n\})$ and define $A^* = \{S \subset \{1, \dots, n\} : S \cap A \neq \varnothing\}$ and B^* similarly. Suppose that $A^* = B$ and $B^* = A$. Prove that $|A| + |B| = 2^n$.

(b) Prove that there is some $c > 0$ so that for every n, there are at least $c \cdot n!$ properties P on subsets of $\{1, \dots, n\}$ for which $P^* = P$. What is the true asymptotic on this number?

11.3: Let (X, d) be a connected compact metric space. Prove that there exists $\alpha > 0$ so that for any $x_1, \dots, x_n \in X$ there is some $x \in X$ such that $\frac{d(x,x_1)+\dots+d(x,x_n)}{n} = \alpha$.

11.4: Show that the Hales-Jewett theorem implies van der Waerden's theorem.

11.5: Using the Hales-Jewett theorem, show that the n-dimensional generalization of van der Waerden's theorem is true.

11.6: Using the Hales-Jewett theorem, show that the multi-dimensional Hales-Jewett theorem is true.

11.7: Using the Hales-Jewett theorem, show that the geometric Ramsey's theorem is true.

11.8: Show that if $\bar{d}(A) > \frac{1}{2}$ then A contains x, y, z so that $x + y = z$.

11.9: Show that if $d^*(A) > 0$ then $A - A$ is syndetic.

11.10: Using Ramsey's theorem, show that if a Δ-set is finitely partitioned, one piece contains a Δ-set.

11.11: Show that if $n \in \mathbb{N}$ is fixed and p is a large enough prime, there exist $x, y, z \not\equiv 0 \pmod{p}$ so that $x^n + y^n \equiv z^n \pmod{p}$.

11.12: Check that for any $n \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\bar{d}_\times(A) = \bar{d}_\times(nA)$.

11.13: Show that $(x_n) \subset [0,1]$ is uniformly distributed if and only if for any continuous $f : [0,1] \to \mathbb{R}$, $\frac{1}{N}\sum_{n=1}^{N} f(x_n) \to \int_0^1 f$ as $N \to \infty$.

11.14: Show that a number $x \in [0,1)$ is base-2 normal if and only if $(2^n x)$ is u.d. mod 1.

11.15: Let $Tx = \left\{\frac{1}{x}\right\}$ (the fractional part of $\frac{1}{x}$) if $x \neq 0$, $T0 = 0$. Let $\mu(A) = \frac{1}{\ln 2}\int_A \frac{dx}{x+1}$. Show that $\mu\left(T^{-1}[a,b]\right) = \mu([a,b])$ for any $[a,b] \subset [0,1]$.

11.16: Show that if $\lambda(A) > 0$ then $A - A$ contains an interval $(-\delta, \delta)$ (λ is the Lebesgue measure).

11.17: Using the Lebesgue points-of-density theorem, show that $T_\alpha : x \mapsto x + \alpha$ is ergodic.

11.18: Show that if $\frac{1}{N}\sum_{n=1}^{N} \mu\left([a,b] \cap T^{-n}[a,b]\right) \to [\mu([a,b])]^2$ as $N \to \infty$ for any interval $[a,b]$ then T is ergodic.

11.19: For $A \subset [0,1]$, let $\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty}(b_k - a_k) : A \subset \bigcup_{k=1}^{\infty}(a_k, b_k)\right\}$. Show that for any $(x_n) \in [0,1]$ there is a set A for which $\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_A(x_n) \not\to \mu^*(A)$ as $N \to \infty$.

Week 12

12.1: Show that the upper density of the primes is 0 (using the fact that $\frac{n \log n}{p_n}$ is bounded).

12.2: Show that $\sum_{n=3}^{\infty} \frac{1}{n \log(n) \log(\log(n))} = \infty$.

12.3: Let P be the set of primes. Is it true that $P - 1$ is GP-rich?

12.4: Show that $P - 1$ contains numbers with arbitrarily many divisors.

12.5: Are there infinitely many squarefree numbers in $P - 1$?

12.6: Show that there is a sequence $(r_n) \subset \mathbb{Q} \cap [0,1)$ which is uniformly distributed and so that $\{r_n : n \in \mathbb{N}\} = \mathbb{Q} \cap [0,1)$.

12.7: Show that (x_n) is dense in $[0,1)$ if and only if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ so that $\left(x_{f(n)}\right)$ is uniformly distributed.

12.8: Show that there is no field consisting of triples of real numbers.

12.9: Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving space. Show that if $\forall A \subset X$ with $\mu(A) > 0$, there is an $n \in \mathbb{N}$ so that $\mu\left(A \cap T^{-n} A\right) > 0$, then for almost every $x \in A$, there is an $m \in \mathbb{N}$ so that $T^m x \in A$.

12.10: Show that the following statement is equivalent to Szemerédi's theorem: If $\bar{d}(A) > 0$ then for all $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ so that $\bar{d}\left(A \cap (A - n) \cap \cdots \cap (A - kn)\right) > 0$.

12.11: Show that if a set is Δ^* then it is Syndetic.

12.12: Show that $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if $\frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n} A\right) \to (\mu(A))^2$ for all $A \in \mathcal{B}$.

12.13: Let p and q be ultrafilters identified with the set of subsets of $\mathbb{N}$ which they give measure 1. Define addition on ultrafilters by saying $A \in p + q$ if and only if $\{n : (A - n) \in p\} \in q$. Show that $p + q \in \beta\mathbb{N}$.

12.14: Let $p, q, r \in \beta\mathbb{N}$ with addition defined as above. Show that $(p + q) + r = p + (q + r)$.

12.15: Let $q_n = \{A \subset \mathbb{N} : n \in A\} \in \beta\mathbb{N}$. Show that $q_n + q_m = q_{n+m}$.

12.16: Show that any finite semigroup has an idempotent element (x is idempotent if $x + x = x$).

Week 13

13.1: Show that for any $\epsilon > 0$, there is some $A \subset \mathbb{N}$ with $\bar{d}(A) > 1 - \epsilon$ so that A contains no shift of an IP-set.

13.2: Show that the equation $x + y = 3z$ is not partition regular.

13.3: Show that $\beta\mathbb{Z}$ forms a semigroup with identity such that the only invertible elements are the principal ultrafilters.

13.4: If $R \subset \mathbb{N}$ has the property that for any $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$, there is some $n \in \mathbb{R}$ for which $\bar{d}\left(A \cap (A - n)\right) > 0$, we say R is a *set of combinatorial recurrence*. Show that sets of combinatorial recurrence are partition regular.

13.5: Define $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$. Check that $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for all $n \geq 0$.

Week 15

15.1: Show that if $\bar{d}(A) > 0$ then $|(A - A) \cap FS(10^n)| = \infty$.

15.2: Prove that piecewise syndeticity is partition regular.

15.3: Show that for any $\epsilon > 0$ and $\alpha \notin \mathbb{Q}$, there is some $n \in \mathbb{N}$ so that $\|n^2\alpha\| < \epsilon$ (where $\|x\|$ is the distance from x to the closest integer to x).

15.4: Show that for any $\epsilon > 0$ and $\alpha \notin \mathbb{Q}$, there is some $n \in \mathbb{N}$ so that $\left(n^2\alpha \bmod 1\right) < \epsilon$.

15.5: Let $\alpha \notin \mathbb{Q}$. Show that if for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ so that $\left(n^2\alpha \bmod 1\right) < \epsilon$, then for any $\epsilon > 0$ there is some $n \in \mathbb{N}$ for which $\left(n^2\alpha \bmod 1\right) > 1 - \epsilon$.

15.6: Let A_n be the set of even permutations of $\{1, \ldots, n\}$, and let $A = \bigcup_{n=1}^{\infty}$ be the set of even finite permutations of $\mathbb{N}$. If $E \subset A$, let $\bar{d}(E) = \limsup_{n\to\infty} \frac{|E \cap A_n|}{|A_n|}$. Show that $\bar{d}$ is invariant under a single group operation.

15.7: Let $V_{\mathbb{F}_p} = \bigoplus_{i=1}^{\infty} \mathbb{F}_p$. If $E \subset V_{\mathbb{F}_p}$, let $\bar{d}(E) = \limsup_{n\to\infty} \frac{|E \cap \bigoplus_{i=1}^{n} \mathbb{F}_p|}{p^n}$. Show that $\bar{d}$ is invariant under a single group operation.

15.8: Show that if G is a group which has an analogue of $\bar{d}$ then, whenever $A \subset G$ with $\bar{d}(A) > \frac{1}{2}$, we have $\bar{d}\left(A \cap x^{-1}A\right) > 0$ for all $x \in G$.

15.9: Is the set $\left\{(n, m) \in \mathbb{N}^2 : \gcd(n, m) = 1\right\}$ syndetic?

15.10: Let F be the free semigroup on two symbols. Let $S_n \subset F$ be the set of words of length n. Let B_n be the set of words of length at most n. What is $\lim_{n\to\infty} \frac{|S_n|}{|B_n|}$?

Week 16

16.1: In $\mathbb{R}^2$, the set of extreme points of a convex body is a closed set. Is the same true in $\mathbb{R}^3$?

16.2: Using the IP enhanced version of Szemerédi theorem, show that $\left\{d : \bar{d}\left(E \cap (E - d) \cap \cdots \cap (E - kd)\right) > 0\right\}$ is IP* for any $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$.

16.3: Show that the Hales-Jewett theorem implies the IP enhanced version of van der Waerden's theorem.

My Handout Exercises

1.1: Show that any sequence $(x_n)_{n=1}^{\infty}$ in $[0,1]$ has a subsequence which converges in $[0,1]$.

1.2: Show that for any irrational number α, the sequence $(n\alpha)_{n=1}^{\infty}$ gets arbitrarily close to the integers. More precisely, show that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ for which the distance from $n\alpha$ to the nearest integer is less than ϵ. Can you use your proof to show that there are infinitely many rational numbers $\frac{p}{q}$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$?

1.3: Let E be a set of positive real numbers, and define its sum as $\sum_{x\in E} x := \sup\left\{\sum_{x\in F} x : F \subset E,\ F \text{ is finite}\right\}$. Show that if E is uncountable then $\sum_{x\in E} x = \infty$.

2.1: Show that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

2.2: 15 students join a summer course. Every day, 3 students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

3.1: Show that Theorem 2 is no longer true if we start with a sequence of only mn distinct real numbers (we still request an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$).

****3.2:** Formulate and prove a generalization of Sperner's lemma (to higher dimensions). Use your generalization to derive a proof for Brouwer's fixed point theorem in any dimension.

4.1: Let $\mathbb{S}^1$ be the unit circle and let $g : \mathbb{S}^1 \to \mathbb{R}$ be continuous. Show that $g(x) = g(-x)$ for some $x \in \mathbb{S}^1$. This is a special case of the Borsuk-Ulam theorem.

***4.2:** Let T be a triangulation of B_2 (the closed unit disk in $\mathbb{R}^2$) whose vertices which lie in $\mathbb{S}^1$ (the unit circle) are symmetric about the origin (i.e. if a vertex $v \in V(T) \cap \mathbb{S}^1$ then $-v \in V(T) \cap \mathbb{S}^1$). Note that the vertices in the interior of the unit disk need not be symmetric about the origin. Also note that an arc in the unit circle between two vertices counts as an edge in $E(T)$, and it must be the edge of some triangle in the triangulation. Let $L : V(T) \to \{-1, +1, -2, +2\}$ be a labeling of the vertices in T which is odd on $\mathbb{S}^1$ (i.e. $L(-v) = -L(v)$ for every $v \in V(T) \cap \mathbb{S}^1$). Show that T contains a complementary edge (an edge whose vertices are labeled with $-i$ and $+i$ for $i = 1$ or 2). *Hint:* Consider walking along some path in the dual graph of T. This result is called Tucker's lemma.

****4.3:** Can you generalize Tucker's lemma (to higher dimensions) and then use your generalization to prove a generalization of exercise 4.1 (the Borsuk-Ulam theorem)? *Hint:* If you're having trouble, try using the internet.

Other Student Talk Exercises

James's talk Exercise 1: Prove that no finite field is algebraically closed.

James's talk Exercise 2: Prove that two finite fields of the same order are isomorphic.

James's talk Exercise 3: Prove that $F_p[x]/(m(x))$ is a field.

James's talk Exercise 4: Prove that if F_p embeds in K (K not necessarily finite) then $(a+b)^{p^n} = a^{p^n} + b^{p^n}$ for all $a, b \in K$ and $n \in \mathbb{N}$.

James's talk Exercise 5: Show that if a finite field K has order p^n then $x^{p^n} - x$ has no double roots (and try to do so without factoring it).

James's talk Exercise 6: A set of integers B is called a sum-free set if for any $x, y \in B$, $x + y \notin B$. Prove that for any set of nonzero integers A, there is a sum-free subset $B \subset A$ of size $|B| > \frac{1}{3}|A|$.

James's talk Exercise 7: Prove that no finite field can be totally ordered. (Can this be extended to say they can't be partially ordered?)

James's talk Exercise 8: Prove that the unit group of $\mathbb{Z}/p^n\mathbb{Z}$ is cyclic.

Vatsa's talk exercise 6: A dominating (finite) sequence of X's and Y's is one in which every initial segment has more X's than Y's. Prove the Cycle lemma, which states: given any sequence of m X's and n Y's, there are exactly $m - n$ cyclic shifts of the sequence which give a dominating sequence.

Vatsa's talk exercise 9: A non-crossing partition of $\{1, 2, \ldots, n\}$ is a partition of the set so that given any two sets S_1, S_2 in the partition, if $a, b \in S_1$ and $x, y \in S_2$, $[a, b] \cap [x, y]$ is either $[a, b]$, $[x, y]$, or $\varnothing$. Prove that the number of non-crossing partitions of $\{1, 2, \ldots, n\}$ is C_n, the nth Catalan number.

Jake's talk exercise 1: Show that Kakeya sets contained in an n-ball of diameter $r < 2$ cannot have arbitrarily small volume.

Jake's talk exercise 2: How many different "directions" are there in $\mathbb{F}_p^2$ for p prime? (i.e. distinct one-dimensional subspaces)? Can you generalize to higher dimensions?

Jake's talk exercise 3: Find a set $K \subseteq \mathbb{R}^2$ containing an infinite line in every direction whose complement has infinite area. Can you make the complement dense in $\mathbb{R}^2$? Can you generalize these results to higher dimensions?