Lec 11/9

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Furstenburg's Ergodic Sz thm:

$\forall$ prob space $(X, \mathcal{B}, \mu)$ and any μ-preserving $T: X \longrightarrow X$.

$\forall A \in \mathcal{B}$ w/ $\mu(A) > 0$, $k \in \mathbb{N}$, $\exists n$ s.t. $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0$.

Ex: $\bar{d}(A) > 0 \implies \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}$ s.t. $d(A \cap A-n \cap \cdots \cap A-kn) > 0$ is equivalent to normal Szemeredi theorem.

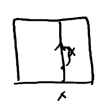
for $k=1$, we have poincare recurrence

Proof: Let $\mu(A) > 0$. Consider $A, T^{-1}A, \ldots, T^{-j}A, \ldots$

$\exists\, i < j$ s.t. $\mu(T^{-i}A \cap T^{-j}A) > 0 \Rightarrow \mu(A \cap T^{-(j-i)}A) > 0$

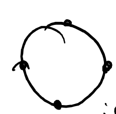
Corollary of proof: $\{n : \mu(A \cap T^{-n}A) > 0\}$ is Δ^* and hence syndetic.

Ex: $\Delta^* \implies$ Syndetic

$(x, y) \longmapsto (x, y + x) \pmod 1$ 

ergodic components in ergodic decomposition:

fibers (α, y) for $\alpha \notin \mathbb{Q}$.

components  (uncountably many)

if $x = \frac{3}{4}$.

Ex: $(X, \mathcal{B}, \mu, T)$ is ergodic iff $\frac{1}{N} \sum \mu(A \cap T^{-n}A) \longrightarrow \mu^2(A)\ \forall A \in \mathcal{B}$

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and now to verify this it suffices to check some generating set of $\mathcal{B}$.

And if $T: x \longmapsto x+\alpha$ then this stuff follows from $(n\alpha)$ is u.d. mod 1.

Defn: An ultrafilter on $\mathbb{N}$ is a 0-1 valued finitely additive prob-measure on $\mathcal{P}(\mathbb{N})$. We identify any ultrafilter with the family of sets having measure 1. $A \in p$ iff $p(A) = 1$. Call A a p-large set.
$\beta\mathbb{N}$ = set of ultrafilters.

given $p, q \in \beta\mathbb{N}$, one defines $p+q$ as follows:

$$A \in p+q \iff \{n : (A-n) \in p\} \in q.$$

Ex: Let $P_n = \{A \subset \mathbb{N} : A \ni n\}$
Show $P_n + P_m = P_{n+m}$.

Ex: ① $p+q \in \beta\mathbb{N}$. ② $(p+q)+r = p+(q+r)$

Fact (Ellis lemma): $\exists\, p \in \beta\mathbb{N}$ s.t. $p+p=p$ (so p is idempotent).

Ex: any finite semigroup has an idempotent element

Let $p = p + p$. Let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ Let $C_i = C \in p$ (i.e. C is p-large).

then $C \in p+p$ so $\{n : (C-n) \in p\} \in p$.

take $n_1 \in C$ s.t. $C \cap C - n_1 \in p$.

take $n_2 \in C \cap C - n_1$ s.t. $(C \cap C - n_1) \cap (C \cap C - n_1) - n_2 \in p$.

$\vdots$

So $FS(n_i) \in C$.