Lec 11/2

Friday, November 2, 2018 14:28

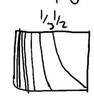
- A function $f : [0,1] \longrightarrow \mathbb{R}$ is $\mathcal{R}$ integrable iff it is bounded & its set of discontinuities is μ-0.

- If $f : \mathbb{R} \to \mathbb{R}$ is monotone then the set of points of nondifferentiability is μ-0.

Ex: T preserves $\mu([a,b]) = \frac{1}{\ln 2}\int_a^b \frac{dx}{x+1}$

$$Tx = \begin{cases} \{\frac{1}{x}\} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

i.e. $\mu(T^{-1}[a,b]) = \mu([a,b])$



"gauss transformation"

- for $x \in [0,1]$, let $x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$ $\quad a_i(x) \in \mathbb{N} \ \forall i.$

$\exists$ K (for Xn+14n+1 (Khintchine), author of book called "Continued fractions")

s.t. for almost every x, $\sqrt[n]{a_1(x)\, a_2(x) \cdots a_n(x)} \longrightarrow K$

- Theorem: (Lebesgue's points of density) Let $A \subset \mathbb{R}$ be a set of positive measure

eg positive measure Cantor set

then for almost every $x \in A$, $\lim_{\varepsilon \to 0} \frac{\lambda(A \cap (x-\varepsilon, x+\varepsilon))}{2\varepsilon} = 1$

cor: $\lambda(A) > 0 \Rightarrow A - A \supset (-\delta, \delta)$ (Ex. hint: it's trivial) (Steinhaus theorem)

(Ergodic) Theorem if $T : [0,1] \longrightarrow [0,1]$ is measure-preserving and ergodic, then $\forall$ "nice" function $f : [0,1] \longrightarrow \mathbb{R}$,

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \xrightarrow{a.e.} \int_0^1 f \, d\mu .$$

(If $\overbrace{T^{-1}A = A}^{\text{invariant sets}}$ a.e. then $\mu(A) = 0$ or 1)

take $f = 1_A$. $\quad \frac{1}{N}\sum_{n=1}^{N} 1_A(T^n x) \xrightarrow[N\to\infty]{a.e.} \int_0^1 1_A \, d\mu = \mu(A)$ if T is ergodic.

eg $T_\alpha : x \longmapsto x+\alpha$ on $\mathbb{T}$.

multiply by 1_B and integrate: $\int_0^1 \frac{1}{N}\sum_{n=1}^{N} 1_B(x) 1_A(T^n x) \, d\mu \xrightarrow[N\to\infty]{} \int_0^1 \mu(A)\, 1_B(x) \, d\mu = \mu(A)\mu(B)$

[can prove it's ergodic using points-of-density thm & some trickery] Ex

$$= \text{Now } \frac{1}{N}\sum_{n=1}^{N} \int_0^1 1_B(x)\, 1_A(T^n x) \, d\mu \quad \xrightarrow[N\to\infty]{a.e.}$$

but $\int_0^1 1_B(x)\, 1_A(T^n x) \, d\mu = \int_0^1 1_B(x)\, 1_{T^{-n}A}(x) \, d\mu$

$$= \mu(B \cap T^{-n} A)$$

So $\frac{1}{N}\sum_{n=1}^{N} \mu(B \cap T^{-n} A) \xrightarrow[N\to\infty]{} \mu(B)\,\mu(A)$

and if this is true $\forall$ measurable A, B,

then T is ergodic $\left(\mu(A \cap T^{-1}A) = \mu(A) \Rightarrow \mu(A) = (\mu(A))^2 \right)$

Example. $Tx = 2x \mod 1$. $f = 1_{[0,\frac{1}{2}]}$

$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$

$$\frac{1}{N}\sum_{n=1}^{N} 1_{[0,\frac{1}{2}]}(2^n x) \xrightarrow{a.e.} \int_0^1 1_{[0,\frac{1}{2}]} = \frac{1}{2}$$

to show $2x \mod 1$ is ergodic it is enough to

Ex

show
$$\frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A) \longrightarrow (\mu(A))^2 \quad \forall \text{ intervals } A = [0, b].$$