Lec 11/16

Friday, November 16, 2018 14:24

- Helly's Theorem
- Caratheodory Theorem

$\}$ from latest packet for midterm.

(read packet for monday after thanksgiving.

$A \subset \mathbb{N}$, $B \subset \mathbb{N}$. something about A & B.

<u>Theorem</u>: $ax + by + cz = 0$ is partition regular iff a subset of its coefficients sums to 0.

i.e. $x + y = 3z$ is <u>not</u> partition regular (<u>Ex</u>)

btw, same is true for $\sum_{i=1}^{n} a_i x_i = 0$.

<u>Rado theorem</u>: google it.

<u>Ex</u>: Prove $\beta\mathbb{Z}$ forms a semigroup w/ identity s.t. the only invertible elements are the principal ultrafilters

Pf of Schur's theorem

first, write $\mathbb{N} = \left(\bigcup_{i=1}^{r_0} C_i\right) \cup \left(\bigcup_{i=r_0+1}^{r} C_i\right)$ s.t. $\bar{d}(C_i) = 0 \;\forall i > r_0$

Second, note that if $\bar{d}(A) = 1$ then A contains a set of differences D.

or, say. sets of the form $\{K, 2K, \dots, 17K\}$.

If $R \subset \mathbb{N}$ is s.t. $\forall A \subset \mathbb{N}$ w/ $\bar{d}(A) > 0$ $\exists n \in R$ s.t. $\bar{d}(A \cap (A-n)) > 0$, we say R is a set of combinatorial recurrence.

Remark: $\{n_i - n_j, i > j\}$ is a set of combinatorial recurrence.

Pf: $\{A - n_i, i \in \mathbb{N}\}$ pigeonhole (since $\bar{d}(A) > 0$)

Let D be a set of differences in $\bigcup_{i=1}^{r_0} C_i$. By partition regularity of Δ-sets, one C_i contains a Δ-set. then we are done

via ramsey's theorem.

Also, $\bar{d}(A \cap (A - (n_i - n_j))) > 0$ so "set of starters of $x, y, x+y$" is big.

Ex: Sets of combinatorial recurrence are partition-regular.

Read: density version of Schur theorem

Let C_n = # of length $2n$ in 2 symbols X & Y s.t. $\#X = \#Y$ and any initial segment has $\#X \geq \#Y$.

$C_3 = 5$:

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$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{n!(n+1)!} = \prod_{k=2}^{n}\left(\frac{n+k}{k}\right)$$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ Ex: Check this!

$$c(X) = \sum_{n=0}^{\infty} C_n X^n \longrightarrow c(X) = 1 + X[c(X)]^2$$

$$c(X) = \frac{1-\sqrt{1-4X}}{2X}$$

$$(1+y)^{1/2} = \sqrt{1+y} = \sum_{n\geq 0} \binom{1/2}{n} y^n$$ (extended binomial thm).

$$= \sum \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \cdots$$

Put $y = -4x$:

$$c(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}$$

Ex: $\forall \varepsilon > 0$, $\exists A \subset \mathbb{N}$ w/ $\bar{d}(A) > 1-\varepsilon$ s.t. A contains no shift of an IP-set.

Solution: an IP-set intersects any $m\mathbb{N}$.

So pick $A = \mathbb{N} \setminus (n_1 \mathbb{N}) \setminus (n_2 \mathbb{N} \pm 1) \setminus (n_3 \mathbb{N} \pm 2) \setminus \cdots$

for some sequence $n_i \nearrow \infty$ quickly enough.