Lec 10/8

Monday, October 8, 2018 14:22

Given: $\bar{d}(A) = 0.99$. Then: $\forall t \in \mathbb{N}$, $\bar{d}(A \cap (A-t)) > 0$.

$A, B \subset \mathbb{N}$ independent if $d(A \cap B) = d(A)\,d(B)$.

Eg: $2\mathbb{N}$ & $3\mathbb{N}$.

Eg: 1_A : 11001100...

1_B : 10101010...

Can you get A s.t. $d(A \cap (A-t)) = d^2(A)$ $\forall t \in \mathbb{N}$.

Let $f : S^1 \xrightarrow[\text{bij.}]{\text{cts}} S^1$ s.t. $\forall x \in S^1$, $\exists n \in \mathbb{N}$ s.t. $f^n(x) = x$.

Ex does $\exists N \in \mathbb{N}$ s.t. $f^N = id$?

(X, d), $f : X \xrightarrow{\text{homeo...}} X$. (X, f) is called (abstract) topological dynamical system.

Def: $(X, f) \approx (Y, g)$ if $\exists h : X \xrightarrow{\text{homeo}} Y$ s.t.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\
Y \arrow[r] & Y
\end{tikzcd}
$$

Commutes

$$\begin{tikzcd} {} \arrow[d, "h"'] & {} \arrow[d] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

$f = h^{-1} g h$ conjugates.

$f^{(n)} = h^{-1} g^{(n)} h$

Call a set $E \subset \mathbb{N}$ normal if 1_E is a normal 0-1 sequence

Claim: if E is normal then $\forall\ 0 < t_1 < t_2 < \cdots < t_k$,

Ex $d(E \cap (E - t_1) \cap (E - t_2) \cap \cdots \cap (E - t_k)) = \frac{1}{2^{k+1}}$

$\Rightarrow \quad x + \{0, t_1, t_2, \ldots, t_k\} \subset E$

E contains a self-shift of any finite set.

Read again: Furstenberg's pf of infinitude of primes

Notions of largeness in $\mathbb{N}$:

1. positive $\bar{d}$ or d^*

2. Syndeticity: finitely many shifts cover $\mathbb{N}$.

Ex: if $\bigcup_{i=1}^{r} (E - t_i) = \mathbb{N}$ then $d^*(E) \geq \frac{1}{r}$.

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Ex: Equivalently, $E \subset \mathbb{N}$ is syndetic if E has bounded gaps.

Ex: If $\bar{d}(E) > 0$ then $E-E$ is syndetic

3. Thick: E contains arbitrarily long intervals. $\iff$ $d^*(E) = 1$

4. AP-rich sets: E contains arbitrarily long APs.
5. GP-rich sets: " " GPs.

GP rich $\not\Rightarrow$ AP rich (2^i)
AP rich $\not\Rightarrow$ GP rich (primes)

If E is syndetic
then $\forall$ long enough interval I, $I \cap E \neq \emptyset$.

6. A set is Δ^*, IP^*, ... if it has non-empty intersection w/ any Δ-set or IP-set.

(A Δ-set is any of the form $\{n_i - n_j : i > j\}$ for some $(n_i)_{i=1}^{\infty}$ w/ $n_i \nearrow \infty$)

if a Δ set is colored $\{n_i - n_j\}_{i>j} = \bigcup_{i=1}^{r} C_i$, one C_i contains a Δ set.

Pf: ramsey's Theorem. induced coloring on pairs $\{i,j\}$.

Can you solve $x+y+z=w$ in any Δ set?

yes.

yes.

T/F: any Δ^* set is cofinite:

No. $2\mathbb{N}$ is Δ^*, but not cofinite

List 17 dangerous theorems for Weds

Review Handout Theorems heavy.

$n\alpha \mod 1$

$$S = \{ n : 0 < n\alpha \mod 1 < \varepsilon \quad \text{or} \quad 1-\varepsilon < n\alpha \mod 1 < 1 \}$$

Ex: S is Δ^*.

Ex: Let $\bar{d}(E) > 0$. Let $n_i \uparrow \infty$.

Then $\exists\ i < j$ s.t. $(E - n_i) \cap (E - n_j) \neq \emptyset$

Corollary: $E - E \ni n_i - n_j$

$\Rightarrow E - E$ is Δ^*

Ex: If E_1, E_2 are Δ^*, so is $E_1 \cap E_2$