

Given:  $\bar{d}(A) = 0.99$ . Then:  $\forall t \in \mathbb{N}, \bar{d}(A \cap (A-t)) > 0$ .

$A, B \subset \mathbb{N}$  independent if  $d(A \cap B) = d(A)d(B)$ .

Eg:  $2\mathbb{N}$  &  $3\mathbb{N}$ .

Eg:  $1_A: 11001100\dots$

$1_B: 10101010\dots$

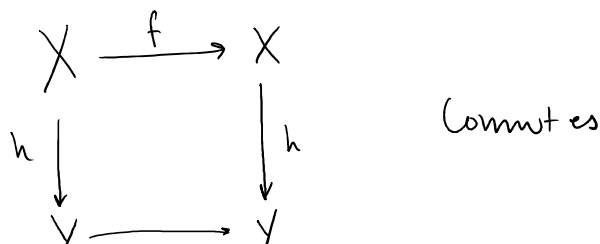
Can you get  $A$  s.t.  $d(A \cap (A-t)) = d^2(A) \quad \forall t \in \mathbb{N}$ .

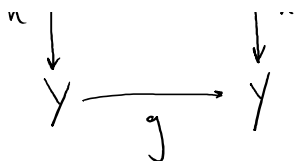
Let  $f: S \xrightarrow[\text{bij.}]{\text{cts}} S'$  s.t.  $\forall x \in S', \exists n \in \mathbb{N}$  s.t.  $f^n(x) = x$ .

Ex  $\uparrow$  does  $\exists n \in \mathbb{N}$  s.t.  $f^n = \text{id}$ ?

$(X, d)$ ,  $f: X \xrightarrow{\text{homeo.}} X$ .  $(X, f)$  is called (abstract) topological dynamical system.

Def:  $(X, f) \approx (Y, g)$  if  $\exists h: X \xrightarrow{\text{homeo.}} Y$  s.t.





$$f = h^{-1} g h \quad \text{conjugates.}$$

$$f^{(n)} = h^{-1} g^{(n)} h$$

Call a set  $E \subset \mathbb{N}$  normal if  $1_E$  is a normal 0-1 sequence

Claim: If  $E$  is normal then  $\forall 0 < t_1 < t_2 < \dots < t_k$ ,

Ex  $\nearrow d(E \cap (E - t_1) \cap (E - t_2) \cap \dots \cap (E - t_k)) = \frac{1}{2^{k+1}}$

$$\Rightarrow X + \{0, t_1, t_2, \dots, t_k\} \subset E$$

$E$  contains a self-shift of any finite set.

Read again: Furstenberg's pf of infinitude of primes

Notions of largeness in  $\mathbb{N}$ :

1. positive  $\bar{d}$  or  $d^*$

2. Syndeticity: finitely many shifts cover  $\mathbb{N}$ .

$\left( \text{Ex: if } \bigcup_{i=1}^r (E - t_i) = \mathbb{N} \text{ then } d^*(E) \geq \frac{1}{r}. \right.$

Ex: if  $\bigcup_{i=1}^r (E - t_i) = N$  then  $d^*(E) > \frac{1}{r}$ .

Ex: Equivalently,  $E \subset N$  is syndetic if  $E$  has bounded gaps.

Ex: If  $d(E) > 0$  then  $E - E$  is syndetic

3. Thick:  $E$  contains arbitrarily long intervals.  $\iff d^*(E) = 1$

4. AP-rich sets:  $E$  contains arbitrarily long APs.

5. GP-rich sets: "

' GPs. GP rich  $\not\equiv$  AP rich (2')  
AP rich  $\not\equiv$  GP rich (primes)

If  $E$  is syndetic

then  $\forall$  long enough interval  $I$ ,  $I \cap E \neq \emptyset$ .

6. A set is  $\Delta^*$ ,  $IP^*$ , ... if it has non-empty intersection w/ any  $\Delta$ -set or  $IP$ -set.

(A  $\Delta$ -set is any of the form  $\{n_i - n_j : i > j\}$  for some  $(n_i)_{i=1}^\infty$  w/  $n_i \nearrow \infty$ )

if a  $\Delta$  set is colored  $\{n_i - n_j : i > j\} = \bigcup_{i=1}^r C_i$ , one  $C_i$  contains a  $\Delta$  set.

Pf: Ramsey's Theorem. induced coloring on pairs  $\{i, j\}$ .

Can you solve  $x + y + z = w$  in any  $\Delta$  set?

yes.

yes .

T/F : any  $\Delta^*$  set is cofinite:

No.  $2\mathbb{N}$  is  $\Delta^*$ , but not cofinite

List 17 dangerous theorems for Weds

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Review Handout Theorems heavy.

$n\alpha \bmod 1$

$$S = \{ n : 0 < n\alpha \bmod 1 < \varepsilon \text{ or } 1 - \varepsilon < n\alpha \bmod 1 < 1 \}$$

Ex :  $S$  is  $\Delta^*$ .

Ex: let  $\delta(\varepsilon) > 0$ . let  $n_i \uparrow \infty$ .

then  $\exists i < j$  s.t.  $(E - n_i) \cap (E - n_j) \neq \emptyset$

Corollary:  $E - E \ni n_i - n_j$

$\Rightarrow E - E$  is  $\Delta^*$

Ex: If  $E_1, E_2 \in \Delta^*$ , so is  $E_1 \cap E_2$