Lec 10/31

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Ramsey Theory: Graham et al

Fact: Let $A \subset \mathbb{N}$, $d^*(A) > 0$. then $\forall k$, the set of differences of APs in A of length K is IP^*. In particular, it is syndetic.

Three principles of ramsey theory:

1. Well organized structures are not destroyable by finite partitions.
2. There is always a notion of largeness behind a partition result.

   $x+y=z$ Schur equation is partition regular.

   Ex: If $\bar{d}(A) > \frac{1}{2}$, then A contains x,y,z s.t. $x+y=z$.

   $\hookrightarrow$ hint: $A \cap A-x \neq \emptyset$ $\forall x$ so $A-A=\mathbb{N}$.

3. "Good" configurations which are present in very large wellspread sets are abundant.

   Ex: if $d^*(A) > 0$, $A-A$ is syndetic (actually, it is Δ^*)

Ex: An application of ramsey's theorem:

If a Δ-set is finitely partitioned, one piece contains a Δ-set.

Note: set of differences is IP^* but not Δ^*

Recall: $\{n : \|n^2\alpha\| < \varepsilon\}$ is IP^* but not Δ^*.

Ex: (Schur, 1916) Theorem: if $n \in \mathbb{N}$ is fixed & p: prime is large enough, $\exists x,y,z$, all not 0 mod p, $x^n + y^n \equiv z^n$ mod p.

Hint: use finitistic version of partition-regularity of $x+y=z$.

take multiplicative subgroup of powers of n,
look at cosets of this subgroup.

Brouwer's Theorem: (Joint extension of vdW & schur.)

$\forall$ finite coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one C_i contains, $\forall k$, configurations of the form $\{z, y, y+z, y+2z, \ldots, y+kz\}$.

Geometric Ramsey's Theorem: If $\bigoplus \mathbb{Z}/p\mathbb{Z} = V_{\mathbb{F}_p} = \bigcup_{i=1}^{r} C_i$ then one of C_i contains arbitrarily large affine subspaces

Some theorems which follow easily from hales-jewett. (all exercises)

1. vdW (with IP^*-ness of set of d).
2. geometric ramsey's theorem
3. $\mathbb{Z}^d$ vdW
4. "Combinatorial" planes, spaces, etc. (multi-dim HJ $\Rightarrow$ Multi-dim vdW).

$$\{ W(t_1, t_2) : t_1, t_2 \in A \}$$

What is d^* in $V_{\mathbb{F}_p}$? Compare to d^*_X in $(\mathbb{N}, X)$

Ex: check asymptotic invariance of $\bar{d}_X$ by multiplicative shifts

x

A sequence $(X_n) \subset [0,1]$ is uniformly distributed if

$\forall\ (a,b) \subset [0,1]$, $\dfrac{\#\{1 \le n \le N : x_n \in (a,b)\}}{N} \longrightarrow b-a$.

ex: Examples: rational #s by increasing denominators

$n\alpha \bmod 1$, $n^2\alpha \bmod 1$, $n^c \bmod 1$, $c>0$, $c \notin \mathbb{Z}$,

$n \log^2 n \bmod 1$.

In fact "almost all" sequences are u.d.

$\forall f \in C[0,1]$, $\frac{1}{N}\sum_{n=1}^{N} f(x_n) \xrightarrow[N\to\infty]{} \int_0^1 f(x)\,dx$

Ex: Theorem: a number $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$, $x_n \in \{0,1\}$ is base-2 normal (i.e. (x_n) is binary normal sequence)

iff $2^n x$ is u.d. mod 1.