# Lec 10/3

Wednesday, October 3, 2018 14:23

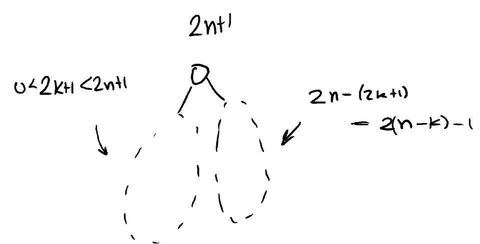
$n^{th}$ Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Ex: if $n=2k$, there is no binary tree w/ $n$ vertices.

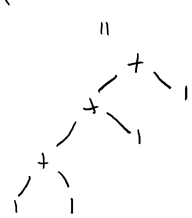
Ex: prove left & right subtrees are complete.

$T_n$ is # of complete binary trees w/ $2n+1$ vertices




$$T_n = \sum_{k=0}^{n-1} T_k T_{n-k-1}$$

$((1+1)+1)+1)$




Ex: Show that # parenthesisations of $1+1+1+1=4$ is $T_3$.
Can this be generalized.

Dyck words

XXYY
XYXY

Ex: Prove that every Dyck word is $X D_1 Y D_2$

Ex. what is base case for Dyck word $\longrightarrow$ binary tree algorithm

A sequence in $x$ and $y$ is dominating if $\# X > \# Y$ at every point.
eg: $Y$, $XY$.

Cycle Lemma: Given any sequence of $m$ $X$'s & $n$ $Y$'s, $\exists$ exactly $m-n$ cyclic shifts of the sequence which give a dominating sequence.

Ex: prove this.

we know $T_n = \#$ Dyck words of length $2n$.

Start w/ dominating seq on $n+1$ $X$'s, $n$ $Y$'s.

must have 2 initial
X X Y X Y
remove to get a dyck word.

$\binom{5}{3}$ seqs on 3 $X$'s, 2 $Y$'s

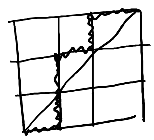
each shifts uniquely to a dyck word.

$$\Rightarrow T_n = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

define $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$, $C_0 = 1$.

Catalan numbers

Applications:



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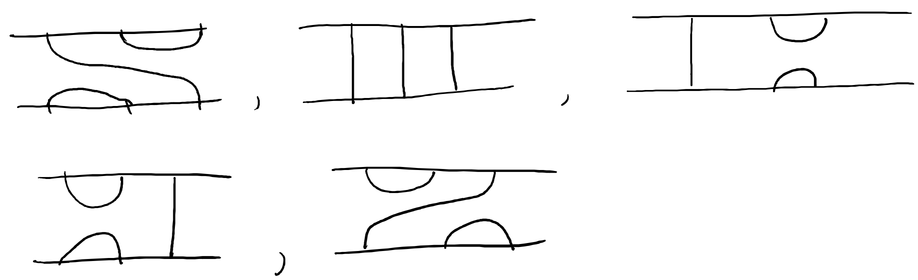
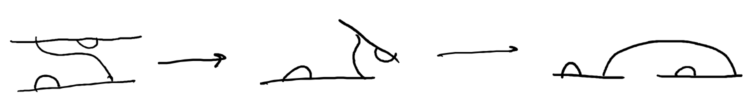
Ex: what is # of paths from $(1,1)$ to $(n,n)$ that don't cross diagonal?

Ex: prove # of Temperly-Lieb diagrams w/ $n$ dots on either side is $C_n$



hint :



Non-crossing partition of $\{1,\dots,n\}$ is a partition of $\{1,\dots,n\}$ s.t. given any two sets $S_1, S_2$ in the partition, if $a,b \in S_1$, $x,y \in S_2$, $[a,b] \cap [x,y] = \begin{cases} [a,b] \\ \emptyset \\ [x,y] \end{cases}$.

Ex: prove that # of non-crossing partitions on $\{1,\dots,n\}$ is $C_n$

Ex: prove that # of non-crossing partitions on $\{1, \ldots, 2n\}$ where each partition has size 2 is $C_n$.